Physics 411: Midterm 2

This is a take-home exam. You have 2 days to complete it. Your answers must be handed in at or before the end of class, 11:30am on Thursday, March 13 to receive a grade. **Answers turned in late will not receive a grade.**

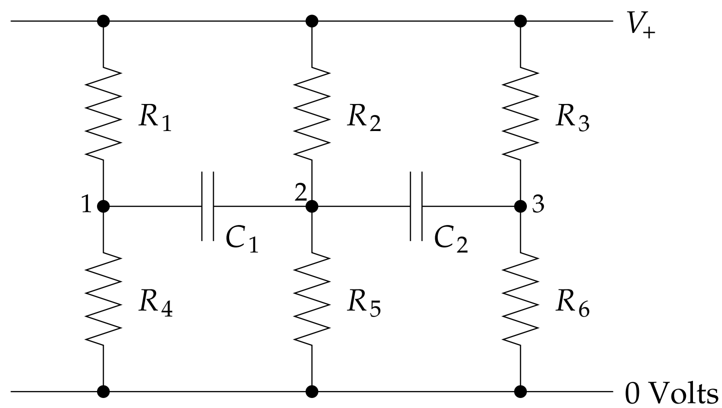
There are four questions, arranged roughly in order of increasing difficulty—the later ones are harder, and also carry more points. Partial credit will be given for all parts of a question that are correctly completed.

The rules are the same as for the previous midterm. You may consult the textbook and you can use any of the on-line resources that accompany the textbook; you may use programs and functions from the on-line resources page as a starting point if you want, as well as programs you yourself have written for the homework assignments. But collaboration is not allowed. You may not discuss the exam with anyone else and you may not look up answers or use material from any other source, including friends or classmates, the internet, or other books. Everything you turn in must be your own original work and no one else's.

Paragraphs marked with a check-mark thus ☑ indicate what is to be handed in for each question.

Good luck!

1. **[6 points]** Here is a circuit utilizing both resistors and capacitors:



The voltage V_+ is time-varying and sinusoidal of the form $V_+ = x_+ e^{i\omega t}$ with x_+ a constant. The resistors in the circuit can be treated using Ohm's law as usual. For the capacitors the charge Q and voltage V across them are related by the capacitor law $Q = CV$, where C is the capacitance. Differentiating both sides of this expression gives the current I flowing in on one side of the capacitor and out on the other:

$$I = \frac{dQ}{dt} = C\frac{dV}{dt}.$$

(a) Assuming the voltages at the points labeled 1, 2, and 3 are of the form $V_1 = x_1 e^{i\omega t}$, $V_2 = x_2 e^{i\omega t}$, and $V_3 = x_3 e^{i\omega t}$, apply Kirchhoff's law at each of the three points, along with Ohm's law and the capacitor law, to show that the constants x_1, x_2, and x_3 satisfy the equations

$$\begin{aligned} \left(\frac{1}{R_1} + \frac{1}{R_4} + i\omega C_1\right)x_1 - i\omega C_1 x_2 &= \frac{x_+}{R_1}, \\ -i\omega C_1 x_1 + \left(\frac{1}{R_2} + \frac{1}{R_5} + i\omega C_1 + i\omega C_2\right)x_2 - i\omega C_2 x_3 &= \frac{x_+}{R_2}, \\ -i\omega C_2 x_2 + \left(\frac{1}{R_3} + \frac{1}{R_6} + i\omega C_2\right)x_3 &= \frac{x_+}{R_3}. \end{aligned}$$

(b) Write a program to solve for x_1, x_2, and x_3 when

$$\begin{aligned} &R_1 = R_3 = R_5 = 1\,\mathrm{k}\Omega, \\ &R_2 = R_4 = R_6 = 2\,\mathrm{k}\Omega, \\ &C_1 = 1\,\mu\mathrm{F}, \qquad C_2 = 0.5\,\mu\mathrm{F}, \\ &x_+ = 3\,\mathrm{V}, \qquad \omega = 1000\,\mathrm{s}^{-1}. \end{aligned}$$

Have your program calculate and print the amplitudes of the three voltages V_1, V_2, and V_3 and their phases in degrees.

Hint 1: Notice that the matrix for this problem has complex elements, but you can still use the `solve` function to solve the equations—it works with either real or complex arguments. Hint 2: You may find the functions `polar` or `phase` in the `cmath` package useful.

If `z` is a complex number then "`r,theta = polar(z)`" will return the modulus and phase (in radians) of `z` and "`theta = phase(z)`" will return the phase alone.

☑ **For full credit** turn your derivations from part (a), and a printout of your program and its output, showing clearly the results you get for the three voltages and phases.

2. **[8 points]** This question asks you to find the roots of the polynomial

$$P(x) = 924x^6 - 2772x^5 + 3150x^4 - 1680x^3 + 420x^2 - 42x + 1.$$

There is no general formula for the roots of a sixth-order polynomial, but one can find them easily enough using a computer.

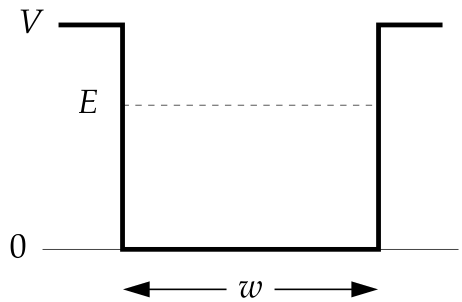
(a) Make a plot of $P(x)$ from $x = 0$ to $x = 1$ and by inspecting it find rough values for the six roots of the polynomial—the points at which the function is zero.

(b) Write a Python program to solve for the positions of all six roots to at least ten decimal places of accuracy, using Newton's method.

Note that the polynomial in this example is just the sixth Legendre polynomial (mapped onto the interval from zero to one), so the calculation performed here is the same as finding the integration points for 6-point Gaussian quadrature, and indeed Newton's method is the method of choice for calculating Gaussian quadrature points.

☑ **For full credit** turn your plot from part (a) along with your rough estimates of the positions of the roots, and a printout of your program from part (b) and its output, showing clearly the final results you get for the roots.

3. **[10 points]** Consider a square potential well of width w, with walls of height V:



Using Schrödinger's equation, it can be shown that the allowed energies E of a single quantum particle of mass m trapped in the well are solutions of

$$\tan\sqrt{w^2 mE/2\hbar^2} = \begin{cases} \sqrt{(V-E)/E} & \text{for the even numbered states,} \\ -\sqrt{E/(V-E)} & \text{for the odd numbered states,} \end{cases}$$

where the states are numbered starting from 0, with the ground state being state 0, the first excited state being state 1, and so forth.

(a) For an electron (mass 9.1094×10^{-31} kg) in a well with $V = 20\,\text{eV}$ and $w = 1\,\text{nm}$, write a Python program to plot the three quantities

$$y_1 = \tan\sqrt{w^2 mE/2\hbar^2}, \qquad y_2 = \sqrt{\frac{V-E}{E}}, \qquad y_3 = -\sqrt{\frac{E}{V-E}}$$

on the same graph, as a function of E from $E = 0$ to $E = 20\,\text{eV}$. From your plot make approximate estimates of the energies of the first six energy levels of the particle.

(b) Write a second program to calculate the values of the first six energy levels in electron volts to an accuracy of 0.001 eV using binary search.

☑ **For full credit** turn your plot from part (a) along with your rough estimates of the positions of the six energy levels, and a printout of your program from part (b) and its output, showing clearly the results you get for the energy levels.

4. **[12 points]** The *variational method* is a method for placing (upper) bounds on the ground-state energy of a quantum system. It involves guessing a form for the ground-state wavefunction of the system and then minimizing the energy of that wavefunction. Since the true ground state is the lowest-energy state of the system, no other state can be lower. Thus any energy you find for your guessed wavefunction must be higher than (or the same as) the ground-state energy. In other words, the method allows you to calculate a number E_{var} such that the true ground-state energy is definitely less than or equal to E_{var}, and you do this without ever solving the Schrödinger equation.

   As an example, consider the quantum simple-harmonic oscillator, meaning a single particle of mass m in a one-dimensional quadratic potential well, and let us guess a possible form for the ground-state wavefunction. Since the system is left-right symmetric and has its lowest potential energy at $x = 0$, the wavefunction should also be symmetric and should be peaked in the middle. We will guess a Gaussian:

$$\psi(x) = \frac{1}{\sqrt[4]{\pi\sigma^2}}\, \mathrm{e}^{-x^2/2\sigma^2},$$

   where σ is the width of the Gaussian and the initial numerical factor ensures that the wavefunction is normalized to unity between $-\infty$ and ∞, so that $\int_{-\infty}^{\infty} |\psi(x)|^2 \, \mathrm{d}x = 1$.

   (a) Using your knowledge of quantum mechanics, show that the expected energy of a particle of mass m with this wavefunction in potential $V(x)$ is

$$\langle E \rangle = \frac{1}{\sqrt{\pi\sigma^2}} \int_{-\infty}^{\infty} \left[\frac{\hbar^2}{2m\sigma^2}\left(1 - \frac{x^2}{\sigma^2}\right) + V(x) \right] \mathrm{e}^{-x^2/\sigma^2} \, \mathrm{d}x.$$

   (b) Write a Python function that takes the value of σ as its argument and uses Gaussian quadrature on 100 points to calculate the value of this integral for an electron in the quadratic potential $V(x) = V_0(x/x_0)^2$ with $x_0 = 10^{-10}\,\mathrm{m}$ and $V_0 = 10\,\mathrm{eV}$. You will have to use an appropriate change of variables to perform the integral over the infinite range from $-\infty$ to ∞. (Hint: All the interesting variation in the integrand happens within a few σ of zero, so make sure your change of variables captures this variation.) Use your function to make a plot of $\langle E \rangle$ against σ for values of σ up to $10^{-9}\,\mathrm{m}$. You should find that the curve goes down and then back up again, with a minimum in the middle. Estimate the approximate position of the minimum energy.

   (c) Combine your function with a golden mean search to find the value of σ that gives the minimum energy. Calculate the value of σ to an accuracy of at least $10^{-15}\,\mathrm{m}$, then use this value to calculate the minimum energy itself, in electron volts.

   (d) This minimum energy is an upper bound on the true ground-state energy of the particle. It is known that the exact ground-state energy E_0 of the quantum harmonic oscillator, in the notation used here, is

$$E_0 = \frac{\hbar}{x_0}\sqrt{\frac{V_0}{2m}}.$$

   How does your result compare?

(e) You should find that your program gives exactly the correct ground-state energy in this case (near enough). This is because a Gaussian is actually the correct functional form for the ground state of the quantum harmonic oscillator. Now modify your program to calculate an upper bound on the ground-state energy of the anharmonic oscillator having $V(x) = V_0[(x/x_0)^2 + (x/x_0)^4]$, with the same values of V_0 and x_0 and using the same Gaussian approximation for the wavefunction.

☑ **For full credit** turn in printouts of your plot from part (b), your final program, its output for the test cases in parts (e) and (f), and your results/answers for parts (a), (b), (c), and (d).