Physics 411: Midterm 1

This is a take-home exam. You have 5 days to complete it. Your answers must be handed in at or before the end of class, 11:30am on Tuesday, February 18 to receive a grade. **Answers turned in late will not receive a grade.**

There are four questions, arranged roughly in order of increasing difficulty—the later ones are harder, and also carry more points. Partial credit will be given for all parts of a question that are correctly completed.

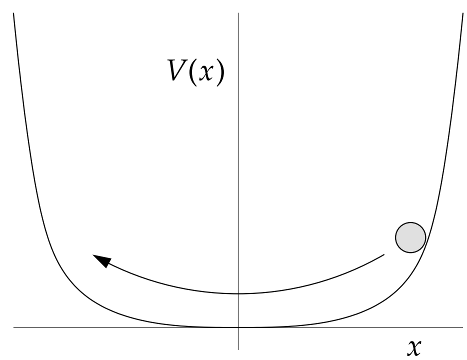
By contrast with the homeworks, collaboration is not allowed on this exam. You must do your work on your own. You may consult the textbook and you can use any of the on-line resources that accompany the textbook; you may use programs and functions from the on-line resources page as a starting point if you want, as well as programs you yourself have written for the homework assignments. But you may not discuss the exam with anyone else and you may not look up answers or use material from any other source, including friends or classmates, the internet, or other books. Everything you turn in must be your own original work and no one else's.

Paragraphs marked with a check-mark thus ☑ indicate what is to be handed in for each question.

Good luck!

1. **[6 points]** The simple harmonic oscillator crops up in many places. Its behavior can be studied readily using analytic methods and it has the important property that its period of oscillation is a constant, independent of its amplitude, making it useful, for instance, for keeping time in watches and clocks. Frequently in physics, however, we also come across anharmonic oscillators, whose period varies with amplitude and whose behavior cannot usually be calculated analytically.

   A general classical oscillator can be thought of as a particle in a concave potential well. When disturbed, the particle rocks back and forth in the well:



   The harmonic oscillator corresponds to a quadratic potential $V(x) \propto x^2$. Any other form gives an anharmonic oscillator. (Thus there are many different kinds of anharmonic oscillator, depending on the exact form of the potential.)

   One way to calculate the motion of an oscillator is to write down the equation for the conservation of energy in the system. If the particle has mass m and position x, then the total energy is equal to the sum of the kinetic and potential energies thus:

$$E = \tfrac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + V(x).$$

   Since the energy must be constant over time, this equation is effectively a (nonlinear) differential equation linking x and t.

   Let us assume that the potential $V(x)$ is symmetric about $x = 0$ and let us set our anharmonic oscillator going with amplitude a. That is, at $t = 0$ we release it from rest at position $x = a$ and it swings back toward the origin. Thus at $t = 0$ we have $\mathrm{d}x/\mathrm{d}t = 0$ and the equation above reads $E = V(a)$, which gives us the total energy of the particle in terms of the amplitude.

   (a) When the particle reaches the origin for the first time, it has gone through one quarter of a period of the oscillator. By rearranging the equation above for $\mathrm{d}x/\mathrm{d}t$ and then integrating with respect to t from 0 to $\frac{1}{4}T$, show that the period T is given by

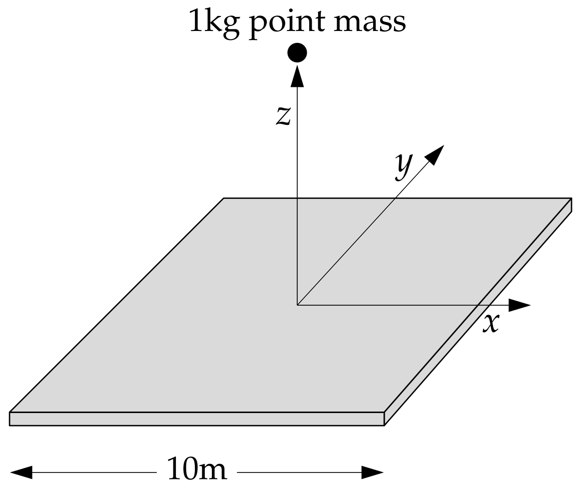
$$T = \sqrt{8m}\int_0^a \frac{\mathrm{d}x}{\sqrt{V(a) - V(x)}}.$$

(b) Suppose the potential is $V(x) = x^2 - \frac{1}{2}x^4$ and the mass of the particle is $m = 1$. Write a Python function that calculates the period of the oscillator for given amplitude a using Gaussian quadrature with $N = 20$ points, then use your function to make a graph of the period for amplitudes ranging from $a = 0$ to $a = 0.99$.

(c) You should find that the period is essentially flat for small amplitudes but gets longer as a approaches one. Explain briefly why this is.

☑ **For full credit** turn in printouts of your program and the plot it produces, plus your answer to part (c) above.

2. **[8 points]** A uniform square sheet of metal is floating motionless in space:



The sheet is 10 m on a side and of negligible thickness, and it has a mass of 10 metric tonnes.

(a) Consider the gravitational force due to the plate felt by a point mass of 1 kg a distance z from the center of the square, in the direction perpendicular to the sheet, as shown above. Show that the component of the force along the z-axis is

$$F_z = G\sigma z \iint_{-L/2}^{L/2} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2+z^2)^{3/2}},$$

where $G = 6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is Newton's gravitational constant and σ is the mass per unit area of the sheet.

(b) Write a program to calculate and plot the force as a function of z from $z = 0$ to $z = 10$ m. For the double integral use (double) Gaussian quadrature, as in Eq. (5.82), with 100 sample points along each axis.

(c) You should see a smooth curve, except at very small values of z, where the force should drop off suddenly to zero. This drop is not a real effect, but an artifact of the way we have done the calculation. Explain briefly where this artifact comes from and suggest a strategy to remove it, or at least to decrease its size.

This calculation can thought of as a model for the gravitational pull of a galaxy. Most of the mass in a spiral galaxy (such as our own Milky Way) lies in a thin plane or disk passing through the galactic center, and the gravitational pull exerted by that plane on bodies outside the galaxy can be calculated by just the methods we have employed here.

☑ **For full credit** turn in printouts of your program and the plot it produces, plus your answer to part (c).

3. **[10 points]** One of the most famous examples of the phenomenon of chaos is the *logistic map*, defined by the equation

$$x' = rx(1-x). \tag{1}$$

For a given value of the constant r you take a value of x—say $x = \frac{1}{2}$—and you feed it into the right-hand side of this equation, which gives you a value of x'. Then you take that value and feed it back in on the right-hand side again, which gives you another value, and so forth. This is a *iterative map*. You keep doing the same operation over and over on your value of x, and one of three things happens:

1) The value settles down to a fixed number and stays there. This is called a *fixed point*. For instance, $x = 0$ is always a fixed point of the logistic map. (You put $x = 0$ on the right-hand side and you get $x' = 0$ on the left.)

2) It doesn't settle down to a single value, but it settles down into a periodic pattern, rotating around a set of values, such as say four values, repeating them in sequence over and over. This is called a *limit cycle*.

3) It goes crazy. It generates a seemingly random sequence of numbers that appear to have no rhyme or reason to them at all. This is *deterministic chaos*. "Chaos" because it really does look chaotic, and "deterministic" because even though the values look random, they're not. They're entirely predictable, because they are given to you by one simple equation. The behavior is *determined*, although it may not look like it.

(a) Write a program that calculates and displays the behavior of the logistic map. Here's what you need to do. For a given value of r, start with $x = \frac{1}{2}$, and iterate the logistic map equation a thousand times. That will give it a chance to settle down to a fixed point or limit cycle if it's going to. Then run for another thousand iterations and plot the points (r, x) on a graph where the horizontal axis is r and the vertical axis is x. You can either use the `plot` function with the options `"ko"` or `"k."` to draw a graph with dots, one for each point, or you can use the `scatter` function to draw a scatter plot (which always uses dots). Repeat the whole calculation for values of r from 1 to 4 in steps of 0.01, plotting the dots for all values of r on the same figure and then finally using the function `show` once to display the complete figure.

Your program should generate a distinctive plot that looks like a tree bent over onto its side. This famous picture is called the *Feigenbaum plot*, after its discoverer Mitchell Feigenbaum, or sometimes the *figtree plot*, a play on the fact that it looks like a tree and Feigenbaum means "figtree" in German.[1]

(b) For a given value of r what would a fixed point look like on the Feigenbaum plot? How about a limit cycle? And what would chaos look like?

[1]There is another approach for computing the Feigenbaum plot, which is neater and faster, making use of Python's ability to perform arithmetic with entire arrays. You could create an array `r` with one element containing each distinct value of r you want to investigate: `[1.0, 1.01, 1.02, ... ]`. Then create another array `x` of the same size to hold the corresponding values of x, which should all be initially set to 0.5. Then an iteration of the logistic map can be performed for all values of r at once with a statement of the form `x = r*x*(1-x)`. Because of the speed with which Python can perform calculations on arrays, this method should be significantly faster than the more basic method above.

(c) Based on your plot, at what value of r does the system move from orderly behavior (fixed points or limit cycles) to chaotic behavior? This point is sometimes called the "edge of chaos."

The logistic map is a very simple mathematical system, but deterministic chaos is seen in many more complex physical systems also, including especially fluid dynamics and the weather. Because of its apparently random nature, the behavior of chaotic systems is difficult to predict and strongly affected by small perturbations in outside conditions. You've probably heard of the classic exemplar of chaos in weather systems, the *butterfly effect*, which was popularized by physicist Edward Lorenz in 1972 when he gave a lecture to the American Association for the Advancement of Science entitled, "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

☑ **For full credit** turn in printouts of your program and the plot it produces, plus your answers to the questions in parts (b) and (c).

4. **[12 points]** A commonly occurring function in physics calculations is the gamma function $\Gamma(a)$, which is defined by the integral

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \, dx.$$

There is no closed-form expression for the gamma function, but one can calculate its value for given a by performing the integral above numerically. You have to be careful how you do it, however, if you wish to get an accurate answer.

(a) Write a program to make a graph of the value of the integrand $x^{a-1}e^{-x}$ as a function of x from $x = 0$ to $x = 5$, with three separate curves for $a = 2$, 3, and 4, all on the same axes. You should find that the integrand starts at zero, rises to a maximum, and then decays again for each curve.

(b) Show analytically that the maximum falls at $x = a - 1$.

(c) Most of the area under the integrand falls near the maximum, so to get an accurate value of the gamma function we need to do a good job of this part of the integral. We can change the integral from 0 to ∞ to one over a finite range from 0 to 1 using the change of variables in Eq. (5.67) in the book, but this tends to squash the peak towards the edge of the $[0, 1]$ range and does a poor job of evaluating the integral accurately. We can do a better job by making a different change of variables that puts the peak in the middle of the integration range, around $\frac{1}{2}$. We will use the change of variables given in Eq. (5.69), which is repeated here for convenience:

$$z = \frac{x}{c + x}.$$

For what value of x does this change of variables give $z = \frac{1}{2}$? Hence what is the appropriate choice of the parameter c that puts the peak of the integrand for the gamma function at $z = \frac{1}{2}$?

(d) Before we can calculate the gamma function, there is another detail we need to attend to. The integrand $x^{a-1}e^{-x}$ can be difficult to evaluate because the factor x^{a-1} can become very large and the factor e^{-x} very small, causing numerical overflow or underflow, or both, for some values of x. Write $x^{a-1} = e^{(a-1)\ln x}$ to derive an alternative expression for the integrand that does not suffer from these problems (or at least not so much). Explain why your new expression is better than the old one.

(e) Now, using the change of variables above and the value of c you have chosen, write a user-defined function `gamma(a)` to calculate the gamma function for arbitrary argument a. Use whatever integration method you feel is appropriate. Test your function by using it to calculate and print the value of $\Gamma(\frac{3}{2})$, which is known to be equal to $\frac{1}{2}\sqrt{\pi} \simeq 0.886$.

(f) For integer values of a it can be shown that $\Gamma(a)$ is equal to the factorial of $a - 1$. Use your Python function to calculate $\Gamma(3)$, $\Gamma(6)$, and $\Gamma(10)$. You should get answers closely equal to $2! = 2$, $5! = 120$, and $9! = 362\,880$.

☑ **For full credit** turn in printouts of your graph from part (a), your final program, its output for the test cases in parts (e) and (f), and your answers to parts (b), (c), and (d).