Complex Systems 535/Physics 508: Homework 6

1. The (wholly fictitious) Northern Gray-Tailed Grebe lives to reproductive age with probability p. If it does so, it always has exactly two offspring. Depending on the value of p, a particular Grebe may have either a finite or an infinite number of descendents in the limit of long time. Let π_s be the probability that the number of a Grebe's descendents is finite and equal to s.

   (i) Show that the generating function for π_s is
   $$h(z) = \frac{1 - \sqrt{1 - 4p(1-p)z^2}}{2pz^2}.$$

   (ii) Hence find the probability u that a Grebe has a finite number of descendents as a function of p.

   (iii) Find the critical value $p = p_c$ below which the Grebe species will always become extinct if we wait long enough.

   (iv) Find an expression for the expected number of descendents a Grebe has when $p < p_c$.

   (v) What is the connection between this problem and networks?

2. Consider a configuration model in which every vertex has the same degree k.

   (i) What is the degree distribution p_k? What are the generating functions g_0 and g_1 for the degree distribution and the excess degree distribution?

   (ii) Show that the giant component fills the whole network for all $k \geq 3$.

   (iii) What happens when $k = 1$?

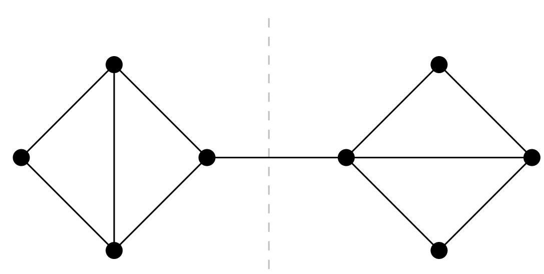
   (iv) **Extra credit:** What happens when $k = 2$?

3. We draw n random reals x in $[0, \infty)$ from the (properly normalized) exponential probability density $P(x) = \mu e^{-\mu x}$.

   (i) Write down the likelihood (i.e., the probability) that we draw a particular set of values x_i (where $i = 1 \ldots n$).

   (ii) Hence find a formula for the best (meaning the maximum-likelihood) estimate of μ given a set of observed values x_i.

4. Consider this small network:



Group 1 Group 2

Suppose we divide it into two groups right down the middle, as indicated by the dotted line, and let us call the group on the left group 1 and the group on the right group 2.

(i) Calculate the (three) quantities m_{rs} and the (two) quantities n_r that appear in the profile likelihood for the two-group stochastic block model (note: just the regular block model, not the degree-corrected variant). Hence calculate the numerical value of the log profile likelihood.

(ii) Verify that no higher profile likelihood can be achieved by moving any single vertex to the other group, and hence that this division into groups is at least a local maximum. (In fact it's the global maximum as well.) Hint: Some of the vertices are symmetry equivalent, which means you need only consider the movement of six different vertices to the other group, which will save you some effort.