1. Suppose we have \( m \) nonnegative integers \( k_1 \ldots k_m \) that are drawn independently, each from its own probability distribution: \( p_1(k_1) \) is the distribution for \( k_1 \), \( p_2(k_2) \) is the distribution for \( k_2 \), and so forth. Let \( g_1(z) \), \( g_2(z) \), etc. be the generating functions for these distributions.

Let \( \pi_s \) be the probability that the sum \( \sum_{i=1}^m k_m \) has the value \( s \). Show that the generating function for \( \pi_s \) is \( h(z) = \prod_i g_i(z) \). Hence, rederive the result we showed in class that the generating function for the sum of \( m \) independent draws from the same distribution \( p(k) \) is the generating function for a single draw raised to the \( m \)th power.

2. Consider the binomial probability distribution

\[
p_k = \binom{n}{k} p^k (1 - p)^{n-k}.
\]

(i) Show that the distribution has probability generating function

\[
g(z) = (pz + 1 - p)^n.
\]

(ii) Find the first and second moments of the distribution from Eq. (13.25) and hence show that the variance of the distribution is \( \sigma^2 = np(1-p) \).

(iii) Show that the sum \( s \) of two numbers drawn independently from the same binomial distribution is distributed according to \( \binom{2n}{s} p^s (1-p)^{2n-s} \).

3. Consider the configuration model with exponential degree distribution

\[
p_k = (1 - e^{-\lambda}) e^{-\lambda k} \quad \text{with } \lambda > 0,
\]

so that the generating functions \( g_0(z) \) and \( g_1(z) \) are given by Eq. (13.130).

(i) Show that the probability \( u \) in Eq. (13.91) satisfies the cubic equation

\[
u^3 - 2e^{\lambda} u^2 + e^{2\lambda} u - (e^{\lambda} - 1)^2 = 0.
\]

(ii) Noting that \( u = 1 \) is always a trivial solution of Eq. (13.91), show that the nontrivial solution corresponding to the existence of a giant component satisfies the quadratic equation

\[
u^2 - (2e^{\lambda} - 1)u + (e^{\lambda} - 1)^2 = 0,
\]

and hence that the size of the giant component, if there is one, is

\[
S = \frac{3}{2} - \sqrt{\frac{e^{\lambda} - 3}{4}}.
\]

Roughly sketch the form of \( S \) as a function of \( \lambda \).

(iii) Show that the giant component exists only if \( \lambda < \ln 3 \).

4. (i) Let us model the Internet as a configuration model with a perfect power-law degree distribution \( p_k \sim k^{-\alpha} \), with \( \alpha \approx 2.5 \) and \( k \geq 1 \). Write down the fundamental generating functions \( g_0 \) and \( g_1 \).

(ii) Hence estimate what fraction of the nodes on the Internet you expect to be functional at any one time (where functional means they belong to the largest component).

5. **Extra credit:** Write a program in the computer language of your choice (or modify your program from last week) to generate a configuration model network in which there are only vertices of degree 1 and 3 and then calculate the size of the largest component.

(i) Use your program to calculate the largest component for a network with \( n = 10000 \) nodes when \( p_1 = 0.6 \) and \( p_3 = 0.4 \) (and \( p_k = 0 \) for all other values of \( k \)).

(ii) Modify your program to make a graph of the size of the largest component for values of \( p_1 \) from 0 to 1 in steps of 0.01. Hence estimate the value of \( p_1 \) for the phase transition at which the giant component disappears.