Complex Systems 535/Physics 508: Midterm Exam

Your name:

Please put your name in the space above.

There are five questions, arranged roughly in order of increasing difficulty. Each question is worth 10 points. You must show your working to get all the points for a question. You will get partial credit for all portions completed correctly.

You can write your answers on this test paper. If you need more space you can attach extra sheets, but please remember to put your name on those as well. The test is open-book, meaning you can refer to the course-pack, notes you took in class, solutions to homework problems, and so forth. You can also use calculators, including graphing calculators.

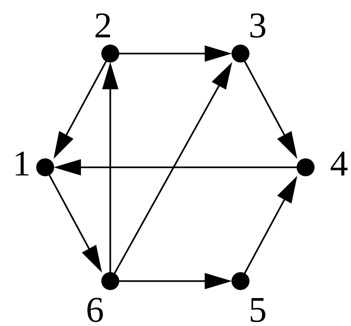
1. **Basic network properties:**

   (a) [**3 points**] Prove that a 3-regular graph must have an even number of vertices.

   (b) [**3 points**] Prove that the average degree of a tree is strictly less than 2.

   (c) [**4 points**] Consider any three vertices in a network; call them A, B, and C. The edge connectivity of A and B is x. The edge connectivity of B and C is y, with $y < x$. What is the edge connectivity of A and C, and why?

2. **Centrality:** Consider this small directed network:



(a) [**2 points**] Is this an acyclic network (and why)?

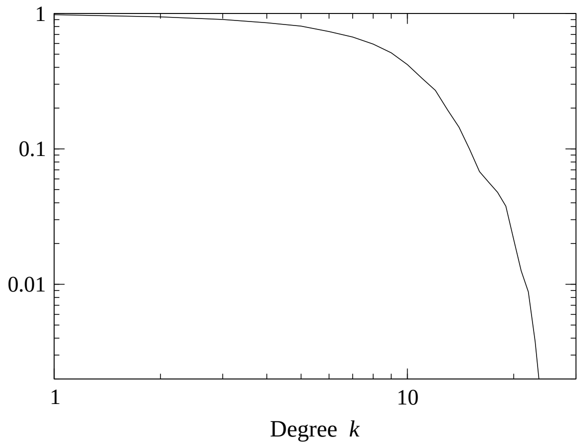
(b) [**2 points**] Write down the adjacency matrix **A** for the network.

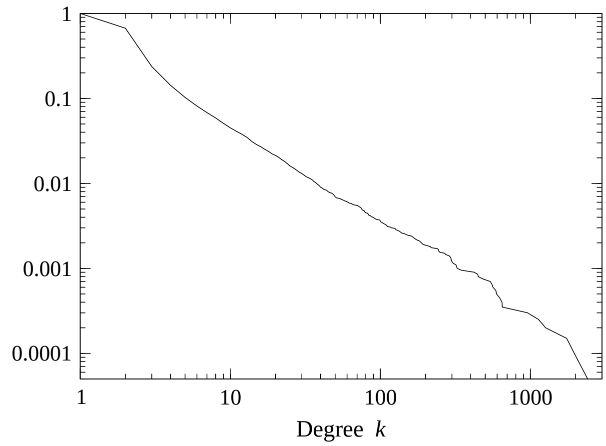
(c) [**2 points**] Hence, or otherwise, find the cocitation matrix.

(d) [**4 points**] Recall that the authority centrality is calculated by multiplying a starting vector repeatedly by the cocitation matrix. Calculate an approximation to the authority centrality for this network by taking the starting vector $(1, 1, 1, \ldots)$ and multiplying it twice by the cocitation matrix.

(e) [**1 point**] Which vertex or vertices have the highest authority centrality in this approximation?

3. **Degree distributions:** Here are plots of the *cumulative* distribution of degrees in two real undirected networks:





(a) [**2 points**] One of these networks, left or right, is approximately scale-free, the other is not. Which is which and how can you tell?

(b) [**2 points**] For the scale-free network give an estimate of the exponent of the degree distribution (which is called α in the book).

(c) [**2 points**] In an infinite scale-free network with this exponent, what value would the second moment $\langle k^2 \rangle$ of the degree distribution take?

(d) [**4 points**] If there are m edges in the scale-free network, then there are $2m$ ends of edges. Approximately what fraction of the highest-degree nodes have a half of all the edge-ends?

4. **Algorithms:** In each of the following you need only describe the algorithm you use in outline. Thus for instance you could say, "Use the power method to calculate the leading eigenvector and then sum the elements," or something of that nature.

   (a) [**2 points**] You are asked to calculate the closeness centrality of a single vertex in a given undirected network with m edges and n vertices. What algorithm would you use to do this, and what would be the time complexity of the operation in terms of m and n?

   (b) [**2 points**] You are given a road map and told the average driving time along each road segment. You are asked to find the route from A to B with the shortest driving time. What algorithm would you use to do this, and what would be the time complexity of the calculation?

   (c) [**2 points**] Recall that a component in an undirected network is a set of vertices with at least one path between every pair. What algorithm would you use to find *all* the components in a network, and what would be the time complexity of the operation?

   (d) [**4 points**] A bicomponent (or 2-component) in a network is a set of vertices with at least two edge-independent paths between every pair. What algorithm would you use to determine whether there are at least two independent paths between a given pair of vertices? Hence or otherwise, suggest an algorithm that can find all the bicomponents in a network.

5. **A random network:** Suppose a network has n vertices, and m undirected edges are strewn uniformly at random between pairs of distinct vertices. That is, for each edge, we pick a starting vertex i uniformly at random and an ending vertex j uniformly at random with $j \neq i$. Then we place the edge between the two and move on to the next edge.

   In the following you can, if you wish, assume that $n^2 \gg m$.

   (a) [**1 points**] What is the mean degree of a vertex in the network?

   (b) [**2 points**] What is the probability that a specific edge will land between two given vertices, i and j, with $i \neq j$? Hence what is the average number of edges we will have between i and j?

   (c) [**3 points**] What is the probability that a specific pair of vertices, i and j, with $i \neq j$, will be connected by a multiedge?

   (d) [**4 points**] What is the probability p_k that a vertex will have degree k?