Complex Systems 535/Physics 508: Homework 8

Because of the Thanksgiving break you have two weeks to complete this homework instead of just one. It is due in class on ***Thursday, December 1****.*

1. Consider a model of a growing directed network similar to Price's model described in Section 14.1, but without preferential attachment. That is, vertices are added one by one to the growing network and each has c outgoing edges, but those edges now attach to existing vertices uniformly at random, without regard for degrees or any other vertex properties.

   (i) Derive master equations, the equivalent of Eqs. (14.7) and (14.8), that govern the distribution of in-degrees q in the limit of large network size.

   (ii) Hence show that in the limit of large size the in-degrees have an exponential distribution $p_q = Ce^{-\lambda q}$ with $\lambda = \ln(1 + 1/c)$.

2. Consider a model network similar to the model of Barabási and Albert described in Section 14.2, in which undirected edges are added between vertices according to a preferential attachment rule, but suppose now that the network does not grow—it starts off with a given number n of vertices and neither gains nor loses any vertices thereafter. In this model, starting with an initial network of n vertices and some specified arrangement of edges, we add at each step one undirected edge between two vertices, both of which are chosen at random in direct proportion to degree k. Let $p_k(m)$ be the fraction of vertices with degree k when the network has m edges in total.

   (i) Show that, when the network has m edges, the probability that the next edge added will attach to vertex i is k_i/m.

   (ii) Write down a master equation giving $p_k(m+1)$ in terms of $p_{k-1}(m)$ and $p_k(m)$. Be sure to give the equation for the special case of $k = 0$ also.

   (iii) Eliminate m from the master equation in favor of the mean degree $c = 2m/n$ and take the limit $n \to \infty$ with c held constant to show that $p_k(c)$ satisfies the differential equation
   $$c\frac{\mathrm{d}p_k}{\mathrm{d}c} = (k-1)p_{k-1} - kp_k.$$

   (iv) Define a generating function $g(c,z) = \sum_{k=0}^{\infty} p_k(c)\, z^k$ and show that it satisfies the partial differential equation
   $$c\frac{\partial g}{\partial c} + z(1-z)\frac{\partial g}{\partial z} = 0.$$

   (v) Show that $g(c,z) = f(c - c/z)$ is a solution of this differential equation, where $f(x)$ is any differentiable function of x.

   (vi) The particular choice of f depends on the initial conditions on the network. Suppose the network starts off in a state where every vertex has degree one, which means $c = 1$ and $g(1,z) = z$. Find the function f that corresponds to this initial condition and hence find $g(c,z)$ for all values of c and z.

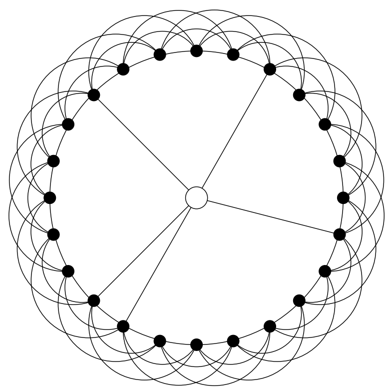
(vii) Show that, for this solution, the degree distribution as a function of c takes the form

$$p_k(c) = \frac{(c-1)^{k-1}}{c^k},$$

except for $k = 0$, for which $p_0(c) = 0$ for all c.

Note that the degree distributions in both this model and the model of question 1 decay exponentially in k, implying that neither preferential attachment alone nor network growth alone can, in general, account for a power-law degree distribution. One must have both growth and preferential attachment to get a power law.

3. A variation on the small-world model that is much easier to solve has been proposed by Dorogovtsev and Mendes. Again we have a ring of n vertices in which each is connected to its c nearest neighbors, where c is even. And again a shortcut is added to the network with probability p for each edge around the ring, but now instead of connecting random vertex pairs, each shortcut connects a random vertex to the same single hub vertex in the center of the network:



This model could be, for example, a model of a (one-dimensional) world connected together by a bus or train (the central vertex) whose stops are represented by the shortcuts.

Show that the mean distance between two vertices in this network in the limit of large n is $\ell = 2(c^2p + 1)/c^2p$ (which is a constant, independent of n).

4. **Extra credit:** Write a computer program in the language of your choice to simulate the vertex degrees in the model of Barabási and Albert using the fast algorithm described in class. (You won't need to store the actual edges of the network itself—only the degrees are needed to answer this question.) Use your program to generate the degree sequence for a Barabási–Albert network of ten million vertices with $c = 2$ and hence make a plot similar to Fig. 14.4 of the average degree of a vertex as a function of time τ, where $\tau = i/n$ for the ith vertex. (You could do this for instance by calculating an average in a sliding window, or simply by dividing the vertices into groups of, say, 1000 each and averaging the degrees within each group.)

Turn in your plot and a copy of your program to get full credit. Extra points may be awarded for particularly elegant programs. (Hint: If you do it right, it should not be a complicated program. In my own program to solve the problem, which is written in C, the main loop that does the actual simulation consists of five lines of code and the program only takes about two seconds to generate a network of ten million vertices.)