# Complex Systems 535/Physics 508: Homework 6

1. Suppose we have $m$ nonnegative integers $k_1 \ldots k_m$ that are drawn independently, each from its own probability distribution: $\pi_1(k_1)$ is the distribution for $k_1$, $\pi_2(k_2)$ is the distribution for $k_2$, and so forth. Let $g_1(z)$, $g_2(z)$, etc. be the generating functions for these distributions.

   Now let $p_s$ be the probability that the sum $\sum_{i=1}^{m} k_m$ has the value $s$. Show that $p_s$ is generated by $h(z) = \prod_i g_i(z)$. Hence, rederive the result we showed in class that the generating function for the sum of $m$ independent draws from the same distribution $\pi(k)$ is the generating function for a single draw raised to the $m$th power.

2. Consider the binomial probability distribution $p_k = \binom{n}{k} p^k (1-p)^{n-k}$.

   (i) Show that the distribution has probability generating function $g(z) = (pz + 1 - p)^n$.

   (ii) Find the first and second moments of the distribution from Eq. (13.25) and hence show that the variance of the distribution is $\sigma^2 = np(1-p)$.

   (iii) Show that the sum of two numbers drawn independently from the same binomial distribution is distributed according to $\binom{2n}{k} p^k (1-p)^{2n-k}$.

3. Consider the configuration model with exponential degree distribution $p_k = (1 - \mathrm{e}^{-\lambda})\mathrm{e}^{-\lambda k}$ with $\lambda > 0$, so that the generating functions $g_0(z)$ and $g_1(z)$ are given by Eq. (13.130).

   (i) Show that the probability $u$ in Eq. (13.91) satisfies the cubic equation $u^3 - 2\mathrm{e}^{\lambda} u^2 + \mathrm{e}^{2\lambda} u - (\mathrm{e}^{\lambda} - 1)^2 = 0$.

   (ii) Noting that $u = 1$ is always a trivial solution of Eq. (13.91), show that the nontrivial solution corresponding to the existence of a giant component satisfies the quadratic equation $u^2 - (2\mathrm{e}^{\lambda} - 1)u + (\mathrm{e}^{\lambda} - 1)^2 = 0$, and hence that the size of the giant component, if there is one, is

   $$S = \tfrac{3}{2} - \sqrt{\mathrm{e}^{\lambda} - \tfrac{3}{4}}.$$

   Roughly sketch the form of $S$ as a function of $\lambda$.

   (iii) Show that the giant component exists only if $\lambda < \ln 3$.

4. The Internet is found to have a power-law degree distribution $p_k \sim k^{-\alpha}$, with $\alpha \simeq 2.5$ and $k \geq 1$.

   (i) Make a mathematical model of the Internet using the configuration model with this degree distribution. Write down the fundamental generating functions $g_0$ and $g_1$.

   (ii) Hence estimate what fraction of the nodes on the Internet you expect to be functional at any one time (where functional means they can actually send data over the network to each other).