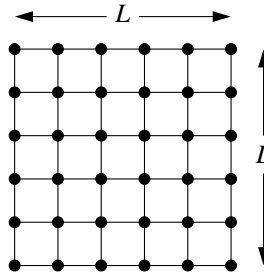


# Complex Systems 535/Physics 508: Homework 3

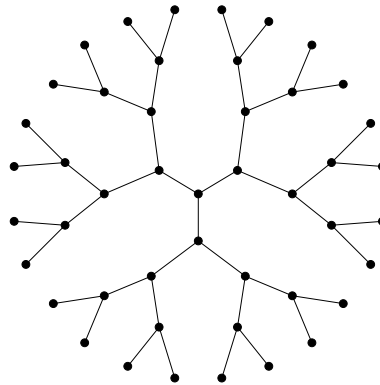
1. One can calculate the diameter of certain types of networks exactly:

- (i) What is the diameter of a clique?
- (ii) What is the diameter of a square portion of square lattice, with  $L$  edges (or equivalently  $L + 1$  vertices) along each side, like this:



What is the diameter of the corresponding hypercubic lattice in  $d$  dimensions with  $L$  edges along each side? Hence what is the diameter of such a lattice as a function of the number  $n$  of vertices?

- (iii) A Cayley tree is a symmetric regular tree in which each vertex is connected to the same number  $k$  of others, until we get out to the leaves, like this:



(We have  $k = 3$  in this picture.)

Show that the number of vertices reachable in  $d$  steps from the central vertex is  $k(k - 1)^{d-1}$  for  $d \geq 1$ . Hence find an expression for the diameter of the network in terms of  $k$  and the number of vertices  $n$ .

- (iv) Which of the networks in parts (i), (ii), and (iii) displays the small-world effect, defined as having a diameter that increases as  $\log n$  or slower?

2. Suppose a network has a degree distribution that follows the exponential form  $p_k = Ce^{-\lambda k}$ , where  $C$  and  $\lambda$  are constants.

- (i) Find  $C$  as a function of  $\lambda$ .
- (ii) Calculate the fraction  $P$  of vertices that have degrees greater than or equal to  $k$ .
- (iii) Calculate the fraction  $W$  of ends of edges that are attached to vertices of degree greater than or equal to  $k$ .

- (iv) Hence show that for this degree distribution the Lorenz curve—the equivalent of Eq. (8.23) in the book—is given by

$$W = P + \frac{1 - e^{-\lambda}}{\lambda} P \ln P.$$

- (v) What is the equivalent of the “80–20” rule for such a network with  $\lambda = 1$ ? That is, what fraction of the “richest” nodes in the network have 80% of the “wealth”?
- (vi) Show that the value of  $W$  is greater than one for some values of  $P$  in the range  $0 \leq P \leq 1$ . What is the meaning of these “unphysical” values?
3. A particular network is believed to have a degree distribution that follows a power law. A random sample of vertices is taken and their degrees measured. The degrees of the first twenty vertices with degrees 10 or greater are:

16	17	10	26	13
14	28	45	10	12
12	10	136	16	25
36	12	14	22	10

Estimate the exponent  $\alpha$  of the power law and the error on that estimate.

4. Consider the following simple and rather unrealistic model of a network: each of  $n$  vertices belongs to one of  $g$  groups. The  $m$ th group has  $n_m$  vertices and each vertex in that group is connected to others in the group with independent probability  $p_m = A(n_m - 1)^{-\beta}$ , where  $A$  and  $\beta$  are constants, but not to any vertices in other groups. Thus this network takes the form of a set of disjoint groups or communities.
- (i) Calculate the expected degree  $\langle k \rangle$  of a vertex in group  $m$ .
- (ii) Calculate the expected value  $\bar{C}_m$  of the local clustering coefficient for vertices in group  $m$ .
- (iii) Hence show that  $\bar{C}_m \propto \langle k \rangle^{-\beta/(1-\beta)}$ . What value would  $\beta$  have to have for the expected value of the local clustering to fall off as  $\langle k \rangle^{-0.75}$ , as has been conjectured by some researchers?