Complex Systems 535/Physics 508: Final Exam sample questions

1. Four small graphs:

   (a) [3 points] Does this graph contain an Eulerian path? (If so, draw the path.) Does it contain a Hamiltonian path? (You don’t need to draw the Hamiltonian path.)

   ![Graph](image)

   (b) [3 points] What is the edge connectivity of vertices $s$ and $t$ in this graph? Draw on the figure to illustrate your answer. What is the vertex connectivity of $s$ and $t$?

   ![Graph](image)

   (c) [2 points] Circle the 2-cliques in this graph:

   ![Graph](image)

   (d) [2 points] What is the local clustering coefficient of vertex A in this graph?

   ![Graph](image)

2. Centrality and algorithms:

   (a) [3 points] Define the degree centrality, closeness centrality, and betweenness centrality of a vertex in an undirected network.

   (b) [3 points] For an undirected graph of $m$ edges and $n$ vertices in adjacency list form, give the leading-order computational complexity, in terms of $m$ and $n$, of the calculation of each of these centralities for a single vertex.
(c) [4 points] Calculate each of these three centralities for the vertex A in the center of this network:

![Diagram of a network with vertex A in the center](image)

3. **Algebraic connectivity:**

(a) [1 point] Write down the definition of the graph Laplacian $L$ of an undirected graph.

(b) [1 point] Write down the definition of the edge incidence matrix $B$ of an undirected graph (the matrix for which $L = B^T B$).

(c) [3 points] Given that $L = B^T B$, show that all eigenvalues of the Laplacian are non-negative.

(d) [2 points] Show that the vector $(1, 1, 1, \ldots)$ is the eigenvector of the Laplacian with the lowest eigenvalue.

(e) [3 points] Hence argue that the algebraic connectivity of an undirected network is zero if the network has more than one component.

4. **Giant component in a random graph:** Consider a random graph with specified degree distribution in the limit of large size. The degree distribution is given by $p_k = C a^k$ for $k \geq 0$, where $0 < a < 1$ and $C$ is a normalizing constant.

(a) [2 points] Find an expression for $C$ in terms of $a$.

(b) [4 points] Find a closed-form expression for the probability generating function for the degree distribution.

(c) [4 points] Hence or otherwise find a condition on $a$—an inequality—that is satisfied if and only if there is a giant component in the network.

5. **Clustering coefficient on a random graph:** A graph of $n$ vertices is constructed as follows. Each vertex $i$ is assigned a “fugacity” $\lambda_i$ in the range $0 \leq \lambda_i \leq 1$, and each vertex pair $(i, j)$ has an edge connecting it with probability proportional to $\lambda_i \lambda_j$.

Calculate the following, in terms of the mean $\langle \lambda \rangle$ and mean square $\langle \lambda^2 \rangle$ of the fugacities, and the number of vertices $n$:

(a) [2 points] The expected degree of vertex $i$.

(b) [3 points] The expected number of triangles in the network for large $n$.

(c) [3 points] The expected number of connected triples for large $n$, i.e., unordered pairs of vertices both connected to another common vertex. (If a particular pair is connected to two other common vertices, count that as two connected triples.)

(d) [2 points] The clustering coefficient for this network in the limit of large $n$. 