Physics 390: The quantum simple harmonic oscillator

The energy eigenstates of the simple harmonic oscillator satisfy $H\psi = E\psi$ with

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2,$$

where $p$ is the quantum momentum operator

$$p = -i\hbar \frac{d}{dx}.$$  

To make things a bit simpler, we change variables to $y = x\sqrt{m\omega/\hbar}$, so that

$$H = -\frac{\hbar}{2m} \frac{\hbar}{\hbar} y^2 + \frac{m\omega}{2} \frac{\hbar}{\hbar} y^2 = -\frac{\hbar\omega}{2} \left[ \frac{d^2}{dy^2} - y^2 \right].$$

Then the Schrödinger equation becomes

$$\left[ \frac{d^2}{dy^2} - y^2 \right] \psi = -\frac{2E}{\hbar^2} \psi.$$  

(4)

Now here’s the trick. Notice that for any function $f(y)$

$$\left[ \left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + y \right) - 1 \right] f = \left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + yf \right) - f$$

$$= \frac{d^2}{dy^2} f + y \frac{d}{dy} f - y \frac{df}{dy} - y^2 f - f$$

$$= \left[ \frac{d^2}{dy^2} - y^2 \right] f.$$  

(5)

Similarly you can show that

$$\left[ \left( \frac{d}{dy} + y \right) \left( \frac{d}{dy} - y \right) + 1 \right] f = \left[ \frac{d^2}{dy^2} + y^2 \right] f.$$  

(6)

Using Eq. (5) we can write Eq. (4) as

$$\left[ \left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + y \right) - 1 \right] \psi = -\frac{2E}{\hbar^2} \psi,$$  

(7)

or equivalently

$$\left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + y \right) \psi = \left( 1 - \frac{2E}{\hbar\omega} \right) \psi.$$  

(8)

Now we operate on both sides with the operator $\frac{d}{dy} + y$ to get

$$\left( \frac{d}{dy} + y \right) \left( \frac{d}{dy} - y \right) \psi = \left( 1 - \frac{2E}{\hbar\omega} \right) \left( \frac{d}{dy} + y \right) \psi,$$  

(9)

and then make use of Eq. (6) to write this as

$$\left[ \frac{d^2}{dy^2} - y^2 - 1 \right] \left( \frac{d}{dy} + y \right) \psi = \left( 1 - \frac{2E}{\hbar\omega} \right) \left( \frac{d}{dy} + y \right) \psi.$$  

(10)

or

$$\left[ \frac{d^2}{dy^2} - y^2 \right] \left( \frac{d}{dy} + y \right) \psi = \left( 2 - \frac{2E}{\hbar\omega} \right) \left( \frac{d}{dy} + y \right) \psi.$$  

(11)

Now we multiply both sides by $-\frac{1}{2}\hbar\omega$ and make use of Eq. (3) again to get

$$H \left( \frac{d}{dy} + y \right) \psi = (E - \hbar\omega) \left( \frac{d}{dy} + y \right) \psi.$$  

(12)

It’s a lot of algebra, but the final result is interesting. If we define a new function $\psi'$ by

$$\psi' = \left( \frac{d}{dy} + y \right) \psi,$$  

(13)

then Eq. (12) can be written as

$$H \psi' = (E - \hbar\omega) \psi'.$$  

(14)

In other words, $\psi'$ is a solution of the Schrödinger equation, but with an energy that’s $\hbar\omega$ less than the energy for $\psi$.

What this means is that if we can find one solution to the Schrödinger equation, with any energy $E$, then we can immediately find another one with energy $E - \hbar\omega$ by applying the operator $\frac{d}{dy} + y$. This operator is called a “lowering operator”—it generates a new solution of lower energy from the old one.
But now we can repeat the process and apply the same operator again to $\psi'$ and get $\psi''$, which has energy $\hbar\omega$ lower still, and so forth. In this way we can get a whole “ladder” of solutions to the Schrödinger equation with energies $\hbar\omega$ apart. In theory we can also go in the opposite direction and increase energy by $\hbar\omega$, extending the ladder in the upward direction as well.

There is, however, a catch. The energy cannot go on decreasing forever: the energy of the simple harmonic oscillator, Eq. (1), is a sum of nonnegative quantities, so it is itself nonnegative. But how can this be, when we have just shown that we can go on decreasing the energy by steps of $\hbar\omega$ for as long as we like? The answer is that at some point there must be a solution $\psi$ of the equation for which the trick above does not work and the operator $d/dy + y$ does not generate another solution with lower energy. How could this happen mathematically? Well, it happens if

\[
\left( \frac{d}{dy} + y \right) \psi = 0.
\]  

(15)

Technically, such a function still satisfies Eq. (12) (because both sides will be zero), but we don’t get a new solution with lower energy.

If we take this equation and operate on both sides with $d/dy - y$ and use Eq. (5) again, we get

\[
\left( \frac{d}{dy} - y \right) \left( \frac{d}{dy} + y \right) \psi = \left[ \frac{d^2}{dy^2} - y^2 + 1 \right] \psi = 0.
\]  

(16)

Then multiplying both sides by $-\frac{1}{2}\hbar\omega$ and making use of (3) we get

\[
H\psi = \frac{1}{2}\hbar\omega \psi.
\]  

(17)

Comparing with the standard Schrödinger equation $H\psi = E\psi$, we see that the state at the bottom of the ladder—the ground state of the simple harmonic oscillator—has energy $E = \frac{1}{2}\hbar\omega$.

Thus, by a process of deduction, we conclude that the energy levels of the simple harmonic oscillator start at $\frac{1}{2}\hbar\omega$ and go up in steps of $\hbar\omega$, so that the $n$th energy level has energy

\[
E_n = (n + \frac{1}{2})\hbar\omega.
\]  

(18)

As a final trick, we can also solve for the actual wave function of the ground state, by solving the differential equation (15). Separating the variables and integrating gives

\[
\int \frac{d\psi}{\psi} = -\int y \, dy,
\]  

(19)

or

\[
\psi = A \exp(-y^2/2) = A \exp(-m\omega x^2/2\hbar),
\]  

(20)

where $A$ is a normalization constant. You can check that this agrees with the answer claimed in the book (Eq. (6-58) in Tipler & Llewellyn).