Physics 406: Summary of important results

This is a list of important equations and other results that you should know. You may take this list into the final exam with you.

Partial derivatives:
\[ df = \frac{\partial f}{\partial x} \bigg|_y dx + \frac{\partial f}{\partial y} \bigg|_x dy, \quad \frac{\partial x}{\partial y} \bigg|_z = \left[ \frac{\partial y}{\partial x} \bigg|_z \right]^{-1}, \quad \frac{\partial x}{\partial y} \bigg|_z = -\frac{\partial x}{\partial z} \frac{\partial z}{\partial y} \bigg|_x. \]

Internal energy:
\[ dU = d\bar{Q} + d\bar{W} \quad \text{with} \quad d\bar{Q} = T \, dS \quad \text{and} \]
\[ d\bar{W} = -p \, dV \quad \text{(fluid pressure/volume system)}, \]
\[ d\bar{W} = f \, dL \quad \text{(spring or wire with force}\ f \text{and length}\ L), \]
\[ d\bar{W} = V \, dq \quad \text{(capacitor with voltage}\ V \text{and charge}\ q), \]
\[ d\bar{W} = \gamma \, dA \quad \text{(surface with surface tension}\ \gamma \text{and area}\ A). \]

Thus for example, in a pressure/volume system \( dU = T \, dS - p \, dV. \) This applies for irreversible as well as reversible changes, but the individual equalities \( d\bar{Q} = T \, dS \) and \( d\bar{W} = -p \, dV \) only apply for reversible ones. Heat capacity at constant \( x \) (where \( x \) is any variable) is in general given by
\[ C_x = T \, \frac{\partial S}{\partial T} \bigg|_x, \quad \text{e.g.,} \quad C_V = T \, \frac{\partial S}{\partial T} \bigg|_V \quad \text{and} \quad C_p = T \, \frac{\partial S}{\partial T} \bigg|_p. \]

Potential functions and Maxwell relations: For a pressure/volume system
\[ H = U + pV \quad \text{(enthalpy)}, \quad F = U - TS \quad \text{(free energy)}, \quad G = U + pV - TS \quad \text{(Gibbs energy)}. \]

Similar expressions apply for other types of systems (non-pressure/volume systems). There is one Maxwell relation for each potential function, derived by equating partial second derivatives. For instance
\[ \frac{\partial^2 U}{\partial V \partial S} = \frac{\partial^2 U}{\partial S \partial V} \Rightarrow \frac{\partial T}{\partial V} \bigg|_S = -\frac{\partial p}{\partial S} \bigg|_V. \]

Each heat capacity is the derivative of the corresponding potential function:
\[ C_V = \frac{\partial U}{\partial T} \bigg|_V, \quad C_p = \frac{\partial H}{\partial T} \bigg|_p. \]

Heat engines:

Efficiency of a reversible engine \( (T_1 > T_2) \):
\[ \eta = \frac{W}{Q_1} = 1 - \frac{T_2}{T_1}, \quad \eta_R = \frac{Q_2}{W} \quad \text{(refrigerator)}, \quad \eta_H = \frac{Q_1}{W} \quad \text{(heat pump)}. \]
Isolated systems: All microstates equally likely. Most likely macrostate maximizes the Boltzmann entropy $S = k \ln g$, where $g$ is the multiplicity. When non-interacting systems are combined, entropy is additive (i.e., extensive); multiplicity is multiplicative.

Fixed temperature systems: States $s$ appear with Boltzmann probability

$$p(s) = \frac{e^{-\varepsilon_s/\tau}}{Z}, \quad Z = \sum_s e^{-\varepsilon_s/\tau},$$

where $\tau = kT$ and $k = 1.38 \times 10^{-23}$ J K$^{-1}$. Macroscopic thermodynamic quantities are then given by

$$U = \tau^2 \frac{\partial \ln Z}{\partial \tau} \bigg|_V, \quad F = -\tau \ln Z, \quad C = \frac{\partial U}{\partial \tau} \bigg|_V, \quad \sigma = -\frac{\partial F}{\partial \tau} \bigg|_V, \quad p = -\frac{\partial F}{\partial V} \bigg|_\tau.$$

Sterling's approximation: $\ln k! \simeq k \ln k - k$.

Ideal gas: Density of states in three dimensions is

$$n(\varepsilon) = \frac{V (2I + 1)}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2},$$

where $I$ is the spin of the particles. ($I = \frac{1}{2}$ for fermions.)

$$Z = \frac{1}{N!} Z_1^N, \quad Z_1 = \frac{V}{(2\pi\hbar^2/m\tau)^{3/2}}, \quad pV = N\tau, \quad \sigma = N \left(\frac{5}{2} - \ln[(2\pi\hbar^2/m\tau)^{3/2}\rho]\right),$$

where $\rho = N/V$ is the number density.

Photons and phonons: Density of states is

$$n(\omega) = \frac{V}{\pi^2 c^3} \omega^2 d\omega \quad \text{(photons, } c \text{ is speed of light)}, \quad n(\omega) = \frac{3V}{2\pi^2 v^3} \omega^2 d\omega \quad \text{(phonons, } v \text{ is speed of sound}).$$

Systems with variable numbers of particles: Grand ensemble:

$$Z = \sum_s e^{-(\varepsilon_s - \mu N_s)/\tau}, \quad U = \tau^2 \frac{\partial \ln Z}{\partial \tau}, \quad \Omega = -\tau \ln Z,$$

$$\sigma = -\frac{\partial \Omega}{\partial \tau} \bigg|_{V, \mu}, \quad p = -\frac{\partial \Omega}{\partial V} \bigg|_{\tau, \mu}, \quad N = -\frac{\partial \Omega}{\partial \mu} \bigg|_{V, \tau}.$$

Quantum gases: Number of particles in single-particle state with energy $\varepsilon$ is

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/\tau} + 1} \quad \text{(fermions)}, \quad f(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/\tau} - 1} \quad \text{(bosons)}.$$