Physics 406: Homework 4

1. Combinatorics: A famous carnival toy is a machine that drops a large number of marbles at the top of a pyramid of nails tacked to a board like this:

The marbles fall down and at each step have a 50% chance of going right or left. At the bottom they fall into a set of bins as shown. If there are \( N \) rows of nails total, it’s pretty easy to convince yourself that there must be \( N + 1 \) bins.

(a) Write down an expression for the number of paths \( g(N, l, r) \) that a marble can take that make a total of \( l \) steps to the left and \( r \) steps to the right. Eliminate \( l \) and \( r \) in favor of the distance \( x = r - l \) traveled to the right to get the same number in terms of \( N \) and \( x \) only. (Note that the distance \( x \) is measured horizontally from where the marbles start, i.e., from a line down the middle of the picture above.)

(b) If \( N = 10 \), how many ways are there of traveling distance \( x = 10 \) to the right? How many ways are there of traveling distance zero?

(c) If \( N = 10 \), about how many marbles will have to be dropped before even a single one of them goes all the way to the right-most bin? And how many if \( N = 20 \)?

(d) When many marbles are dropped, what is the expected mean distance \( \langle x \rangle \) traveled, averaged over all of them? And what is the standard deviation of the distance?

(e) So if you had to say where a single marble dropped would land, between about which values of \( x \) would you feel reasonably confident saying it would end up, if \( N = 100 \)? (If you want to be really precise, you could say which values would you have 90% confidence it would land between, but any sensible answer will do for this question. Saying that \( x \) lies between \(-100 \) and 100 is not a good answer!)

2. Entropy of a set of harmonic oscillators: The quantum simple harmonic oscillator is a quantum system with equally spaced energy levels \( \varepsilon = sh\omega \), where \( h \) and \( \omega \) are constants and \( s \) is a non-negative integer. If we have \( N \) identical such oscillators, their total combined energy can take values \( U = nsh\omega \), where \( n \) is another non-negative integer. In your course pack it is shown that the multiplicity of the state with energy \( nsh\omega \) is given by the binomial coefficient (or combination)

\[
g(N, n) = \binom{N - 1 + n}{N - 1}.
\]
(a) Write \( g(N, n) \) in a form involving factorials and hence write down the dimensionless entropy \( \sigma \) of the system when in thermal isolation.

(b) When \( N \) is large we can, to a good approximation, replace \( N - 1 \) by \( N \). Do this for your expression.
A result which we will use a lot in this course is Sterling’s approximation for the logarithm of a factorial, which says that

\[
\ln n! \simeq n \ln n - n,
\]

where the approximation becomes better and better as \( n \to \infty \). We will see how to prove this result in a later lecture. For the moment we just assume it. (If you want to see the proof, it’s given in Appendix A of Kittel and Kroemer.) Apply Sterling’s approximation to your expression for \( G(N, n) \) and derive an approximate expression for \( \sigma \) for large \( N \).

(c) Recalling the definition of the temperature \( \tau \) in energy units, \( \tau = \partial U / \partial \sigma \), differentiate to get an expression for \( \tau \) in terms of the internal energy. (You have to consider \( n \) to be a continuous variable to do this calculation, which is strictly speaking not correct—it is an integer. Later in the course we’ll see a better derivation of this result that doesn’t require us to do this fix.)

(d) Rearrange to show that

\[
U = \frac{N \hbar \omega}{\exp(\hbar \omega / \tau) - 1}.
\]

This is the internal energy of a set of \( N \) harmonic oscillators and, as we will show, is also the correct expression for the energy of a set of bosons (e.g., photons) in a quantum gas. From this expression we will later derive the famous black-body radiation spectrum of Rayleigh and Planck.

(e) What is the heat capacity of the system?

3. **Partition function of a simple system:** Suppose a simple system has states with three energies, \(-\epsilon, 0, \) and \(+\epsilon\). The multiplicities of the states are \( g(-\epsilon) = 1, g(0) = 2, \) and \( g(\epsilon) = 1 \). The system is put in contact with a thermal reservoir at temperature \( \tau \) (in energy units) and allowed to come to equilibrium.

(a) Calculate the partition function \( Z \) of the system.

(b) Calculate the average internal energy of the system as a function of temperature.

(c) Show that the heat capacity is

\[
C = \frac{\epsilon^2}{\tau^2 \left[ 1 + \cosh(\epsilon / \tau) \right]}.
\]