Complex Systems 899: Math Quiz

These questions will not be graded. They are for your own practice. You are not expect to hand them in. Worked solutions are available from me, Mark Newman, upon request.

These questions are at the level of the most difficult math you'll need to know to complete this course. If you can just barely do them, then you're exactly at the level the course is aimed at. If you think they're easy, you'll ace it. If you can't do them, you might want to come and talk to me.

1. Consider the quantity
$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t}\, \mathrm{d}t.$$
Show that $\Gamma(x) = (x-1)\Gamma(x-1)$ for $x > 0$ and that $\Gamma(1) = 1$. Hence, by induction, show that $\Gamma(n) = (n-1)!$ for n a nonnegative integer (with the convention that $0! = 1$).

   Now show that for general, non-integer ν
$$(1+x)^\nu = \sum_{k=0}^\infty \frac{\Gamma(\nu)}{\Gamma(\nu-k)\Gamma(k+1)} x^k.$$
(Highly rigorous demonstrations are not necessary: you can simply derive the form of the individual coefficients and then show that they are given correctly by the expression above.)

2. Consider an $n \times n$ matrix with all elements real. (a) Is it always possible to diagonalize the matrix? (b) Is it always possible to diagonalize the matrix if it is symmetric? If your answer to either question was "no," give a nondiagonalizable example matrix.

3. Write an expression in terms of factorials for the number of ways $\binom{n}{m}$ of choosing m objects from a set of n, where the order in which you choose them does not matter.

   A jar contains n balls, r of them red and the remaining $n-r$ green. You draw m balls at random from the jar. Show that the probability $P(k)$ that k of them are red is
$$P(k) = \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}.$$
Now, using the fact that the probabilities must sum to 1 derive the identity
$$\sum_{k=0}^m \binom{r}{k}\binom{n-r}{m-k} = \binom{n}{m}.$$
Hence show that the number of red balls drawn from the jar, averaged over many repetitions of the experiment, has the naive expected value of rm/n. (You can do this just by writing down the average and then juggling the factorials until you get something like the form above.)

More on the reverse side →

4. Suppose the population of a country at time t is $x(t)$, and let $x(t)$ be any real positive number. (Obviously population is actually an integer, but if x is large the distinction may not matter much.) The rate $\mathrm{d}x/\mathrm{d}t$ of population growth is equal to c times the size of the population (where c is a constant) plus a term $f(t)$ representing the rate of immigration at time t (or emigration if $f(t)$ is negative).

   Write down the differential equation satisfied by x. Now find an integrating factor for the terms involving x, and hence derive the general expression for the solution of the equation if $x(0) = x_0$. (Your general solution will involve an integral over a function involving f, which you won't be able to do because f hasn't been specified yet.)

   Now give the particular solution for $x(t)$ when $f(t) = at$ with a constant.