Complex Systems 899: Homework 2

For full credit, show all your working.

1. **Transcritical bifurcations:** Show that these systems have transcritical bifurcations and find the values of r at which they occur. Also sketch the bifurcation diagram of x^* against r.

   (a) $\dot{x} = rx - \ln(1 + x)$

   (b) $\dot{x} = x(r - e^x)$

2. **Pitchfork bifurcations:** Show that these systems have pitchfork bifurcations and find the value of r at which they occur. Classify each as supercritical or subcritical.

   (a) $\dot{x} = rx - \sinh x$

   (b) $\dot{x} = x + rx/(x^2 + 1)$

3. **A trickier example:** Consider the system defined by $\dot{x} = rx - \sin x$.

   (a) Find and classify all the fixed points for $r = 0$.

   (b) Show that when $r > 1$ there is only one fixed point.

   (c) As r decreases from ∞ to 0 classify all the bifurcations that occur, in order.

   (d) For $0 < r \ll 1$, find an approximate formula for the values of r at which bifurcations occur.

   (e) Repeat your analysis for $r < 0$ and hence sketch the bifurcation diagram, indicating the stability of the various branches with solid or dotted lines.

4. **Higher order pitchforks:** At a pitchfork bifurcation a single fixed point splits into three—like the prongs of a pitchfork. Give an example of a system $\dot{x} = f(x)$ in which a single fixed point splits into five at $r = 0$. Is it possible to find such a system that splits into four, and if not, why not?

5. **A simple two-dimensional system:** Consider the system $\dot{x} = -y$, $\dot{y} = -x$.

   (a) Show that the trajectories of the system are hyperbolas of the form $x^2 - y^2 = C$ where C is a constant. (Hint: Show first that the equations imply $x\dot{x} - y\dot{y} = 0$.)

   (b) The origin is a saddle point. Find equations for its stable and unstable manifolds.

   (c) Solve exactly for the behavior of the system by making a change of variables to $u = x + y$ and $v = x - y$, with the initial condition $x(0) = x_0$, $y(0) = y_0$.