Complex Systems 899: Homework 1

For full credit, show all your working.

1. **Fixed points:** For each of the following cases, give an equation of the form \( \dot{x} = f(x) \) with the stated properties or, if there is no such equation, explain why not:

   (a) Every integer is a stable fixed point.
   (b) Every real number is a fixed point.
   (c) There are precisely three fixed points, and all of them are stable.
   (d) There are no fixed points.
   (e) There are precisely 100 fixed points.

2. **Exact solution of the logistic equation:** Derive the exact solution of the logistic equation \( \dot{x} = \beta x(1 - x) \), where \( \beta \) is a constant, for the case \( x(0) = x_0 \) with \( x_0 > 0 \).

3. **Nonunique solutions:** The equation \( \dot{x} = \sqrt{x} \) with initial condition \( x(0) = 0 \) has a trivial solution \( x(t) = 0 \) for all \( t \). Find another solution to the same equation with the same boundary. Explain the physical reason why there are two solutions. How would a real system decide which solution to follow?

4. **Linear stability analysis:** Use linear stability analysis to classify the fixed points of the following equations (or show that such an analysis breaks down):

   (a) \( \dot{x} = x(1 - x)(2 - x) \)
   (b) \( \dot{x} = \tan x \)
   (c) \( \dot{x} = x^2(4 - x) \)
   (d) \( \dot{x} = e^{-1/x^2} \)
   (e) \( \dot{x} = \ln x \)

5. **Saddle-node bifurcations:** For each of the following equations show that there is a saddle-node bifurcation in the dynamics of \( x \) for some value of the parameter \( r \), and determine that value. Sketch the bifurcation diagram of fixed points \( x^* \) as a function of \( r \) in each case:

   (a) \( \dot{x} = r - \cosh x \)
   (b) \( \dot{x} = x^2 + rx + 1 \)
   (c) \( \dot{x} = r + x - \ln(x + 1) \)
   (d) \( \dot{x} = r^2 + \frac{1}{4}x - x/(1 + x) \)