Two spin systems in contact: A result which we will use a lot in this course is Sterling’s approximation for the logarithm of a factorial, which says that
\[
\ln n! \approx n \ln n - n,
\]
where the approximation becomes better and better as \( n \to \infty \). We will see how to prove this result in a later lecture. For the moment we just assume it. (If you want to see the proof, it’s given in Appendix A of Kittel and Kroemer.)

Now consider the spin system we looked at, composed of \( N \) “spins” that each point either up or down.

(a) Write down an expression for the multiplicity \( g(N, s) \) of states that have a given value of the spin excess parameter \( s \), defined by \( 2s = N \uparrow - N \downarrow \). Hence write down an expression for the logarithm of the multiplicity as a function of \( s \).

(b) Apply Sterling’s approximation and show that for large \( N \)
\[
\ln g = N \ln N - (\frac{1}{2}N + s) \ln(\frac{1}{2}N + s) - (\frac{1}{2}N - s) \ln(\frac{1}{2}N - s).
\]

(c) Noting that \( \ln(\frac{1}{2}N + s) = \ln(\frac{1}{2}N) + \ln(1 + 2s/N) \), expand to second order in \( 2s/N \) and hence show that
\[
g(N, s) \approx 2^N e^{-2s^2/N}.
\]

This is a Gaussian or normal distribution: the binomial distribution becomes a Gaussian distribution for large \( N \).

(d) Now suppose we have two identical systems of \( N \) spins. Using Eq. (2), write down an expression for the total multiplicity of both systems together as a function of the spin excesses \( s_1 \) and \( s_2 \) of the two systems, assuming for the moment that the two systems are not in contact with one another. (To make the calculations easier, you can assume \( N \) is even.) Eliminate \( s_2 \) in favor of the total spin excess \( s = s_1 + s_2 \) and, by completing the square, show that for a given value of \( s \) the distribution of possible values of \( s_1 \) is Gaussian.

(e) Now suppose that both systems are in a magnetic field of intensity \( B \), so that they have energies \( U_1 = -2mBs_1 \) and \( U_2 = -2mBs_2 \), where \( m \) is the dipole moment of each spin. If the two systems are now in contact with one another, so that energy can flow between them, then \( s_1 \) and \( s_2 \) are no longer fixed, but the total energy must be conserved, so \( s = s_1 + s_2 \) is constant. What is the most likely value of \( s_1 \)?

2. Entropy of a set of harmonic oscillators: Consider the ensemble that we discussed earlier of \( N \) quantum harmonic oscillators. Each can have energy \( \epsilon_s = s\hbar\omega \), where \( \hbar \) and \( \omega \) are constants and \( s \) is a non-negative integer. As we showed in class, the multiplicity \( g(N, n) \) of states of the entire ensemble that have total internal energy \( U = n\hbar\omega \) is given by
\[
g(N, n) = \binom{N - 1 + n}{N - 1}.
\]

(a) Write \( g(N, n) \) in a form involving factorials and hence write down the dimensionless entropy \( \sigma \) of the system.
(b) When $N$ is large we can, to a good approximation, replace $N - 1$ by $N$. Apply Sterling’s approximation and derive an approximate expression for $\sigma$ for large $N$.

(c) Recalling the definition of the temperature $\tau$ in energy units, $\tau = \partial U / \partial \sigma$, differentiate to get an expression for $\tau$ in terms of the internal energy. (You have to consider $n$ to be a continuous variable to do this calculation, which is strictly speaking not correct—it is an integer. Later in the course we’ll see a better derivation of this result that doesn’t require us to do this kludge.)

(d) Rearrange to show that

$$U = \frac{N\hbar \omega}{\exp(h\omega/\tau) - 1}.$$  

This is the internal energy of a set of $N$ harmonic oscillators and, as we will show, is also the correct expression for the energy of a set of bosons (e.g., photons) in a quantum gas. From this expression we will later derive the famous black-body radiation spectrum of Rayleigh and Planck.

(e) What is the heat capacity of the system?

3. **Partition function of a simple system:** Suppose a simple system has states with three energies, $-\epsilon$, $0$, and $+\epsilon$. The multiplicities of the states are $g(-\epsilon) = 1$, $g(0) = 2$, and $g(\epsilon) = 1$. The system is put in contact with a thermal reservoir at temperature $\tau$ (in energy units) and allowed to come to equilibrium.

(a) Calculate the partition function $Z$ of the system.

(b) Calculate the average internal energy of the system as a function of temperature.

(c) Show that the heat capacity is

$$C = \frac{\epsilon^2}{\tau^2 \left[ 1 + \cosh(\epsilon/\tau) \right]}.$$