Complex Systems 535/Physics 508: Homework 8

1. **Bipartite random graphs:** Consider a bipartite network, such as, for instance, the network of movie actors and the movies they appear in. Suppose there are n actors and m movies. The fraction of actors that have appeared in j movies is p_j and the fraction of movies that have k actors in them is q_k. The means of these two distributions are

$$\mu = \sum_{j=0}^{m} j p_j, \qquad \nu = \sum_{k=0}^{n} k q_k.$$

We can model this network with a random bipartite graph having the same degree distributions of the two types of vertices but being otherwise random. We consider the limit where $m, n \to \infty$.

(i) Derive a relationship giving ν in terms of μ, m, and n.

(ii) Let us define generating functions

$$f_0(x) = \sum_j p_j x^j, \qquad g_0(x) = \sum_k q_k x^k.$$

Write down the equivalent functions f_1 and g_1 for the excess degrees of vertices of each type at the end of an edge.

(iii) Let $F_0(x)$ be the generating function for the degree distribution of the 1-mode projection of the network onto the actors alone. Write down an expression for $F_0(x)$ in terms of the fundamental generating functions of the previous part.

(iv) Thus show that the mean degree z_1 of the 1-mode projected network is given by

$$z_1 = \frac{\mu}{\nu} g_0''(1).$$

(v) Let $F_1(x)$ be the generating function for the excess degree distribution of the vertex reached by following an edge on the 1-mode projection. Write down an expression for F_1. (Excess degree in this context means the number of other edges leaving a vertex you just reached *other than* those that lead to other people in the same group that you share with the vertex. In other words it is the obvious thing—the number of people two steps away from you via this vertex.)

(vi) Hence show that the mean number z_2 of vertices two steps away from a randomly chosen vertex on the 1-mode network is

$$z_2 = z_1 f_1'(1) g_1'(1).$$

(vii) Hence argue that there exists a giant component in the 1-mode network if and only if

$$\sum_{jk} jk(jk - j - k) p_j q_k > 0.$$

(viii) Suppose the two degree distributions are Poissonian with means μ and ν. Write down the four basic generating functions and hence find the mean degree of the 1-mode network of actors, and the condition that must be satisfied by μ and ν for there to be a giant component.

2. **Phase transition in a directed graph:** Consider a directed random graph of the kind we discussed in class.

   (i) If the in- and out-degrees of vertices are uncorrelated, show that a giant strongly connected component exists in the graph if and only if $z(z-2) > 0$, where z is the mean out-degree.

   (ii) In real directed graphs the degrees are usually correlated. In the world wide web, for instance, the in- and out-degrees of the vertices have a measured covariance of about $\rho = 180$. The mean degree is $z = 4.6$. Derive a condition for the existence of a giant component as a function of ρ and z, and hence say whether we would expect the web to have a giant strongly connected component on the basis of these measurements.

3. **A growing random graph:** Consider the following simple model of a growing network. Vertices are added to a network at a rate of one per unit time. Edges are added at a mean rate of β per unit time, where β can be anywhere between zero and ∞. (That is, in an small interval Δt of time, the probability of an edge being added is $\beta\Delta t$.) Edges are placed uniformly at random between any pair of vertices that exist at that time. They are never moved after they are first placed.

   We will tackle this model using a rate equation method. Let $a_{k,n}$ be the fraction of vertices that belong to components of size k when there are n vertices in the graph. That is, if we choose a vertex at random from the n vertices currently in the graph, $a_{k,n}$ is the probability the vertex will fall in a component of size k.

   (i) What is the probability that a newly appearing edge will fall between a component of size r and another of size s? (You can assume that n is large and the probability of both ends of an edge falling in the same component is small.) Hence what is the probability that a newly appearing edge will join together two pre-existing components to form a new one of size k?

   (ii) What is the probability that a newly appearing edge joins a component of size k to a component of any other size, thereby creating a new component of size larger than k?

   (iii) Thus write down a rate equation that gives the fraction of vertices $a_{k,n+1}$ in components of size k for $n+1$ vertices, in terms of the values for n vertices.

   (iv) The only exception to the previous result is that components of size 1 appear at a rate of one per unit time. Write a separate rate equation for $a_{1,n+1}$.

   (v) If a steady state solution exists for the component size distribution, show that it must satisfy the equations
   $$(1+2\beta)a_1 = 1, \qquad (1+2\beta k)a_k = \beta k \sum_{j=1}^{k-1} a_j a_{k-j}.$$

   (vi) Multiply by z^k and sum over k from 1 to ∞ and hence show that the generating function $g(z) = \sum_k a_k z^k$ satisfies the ordinary differential equation
   $$2\beta \frac{\mathrm{d}g}{\mathrm{d}z} = \frac{1-g/z}{1-g}.$$

   (vii) **Lots of extra credit:** A humongous number of extra points go to anyone who can find a nontrivial solution to this equation in closed form for the appropriate boundary conditions ($g(0) = 0$). Series expansion solutions count if the series coefficients are in closed form, or solutions making use of special functions. (I should point out that I don't know of any solution to this equation, but I don't claim to be great at solving nonlinear equations.)