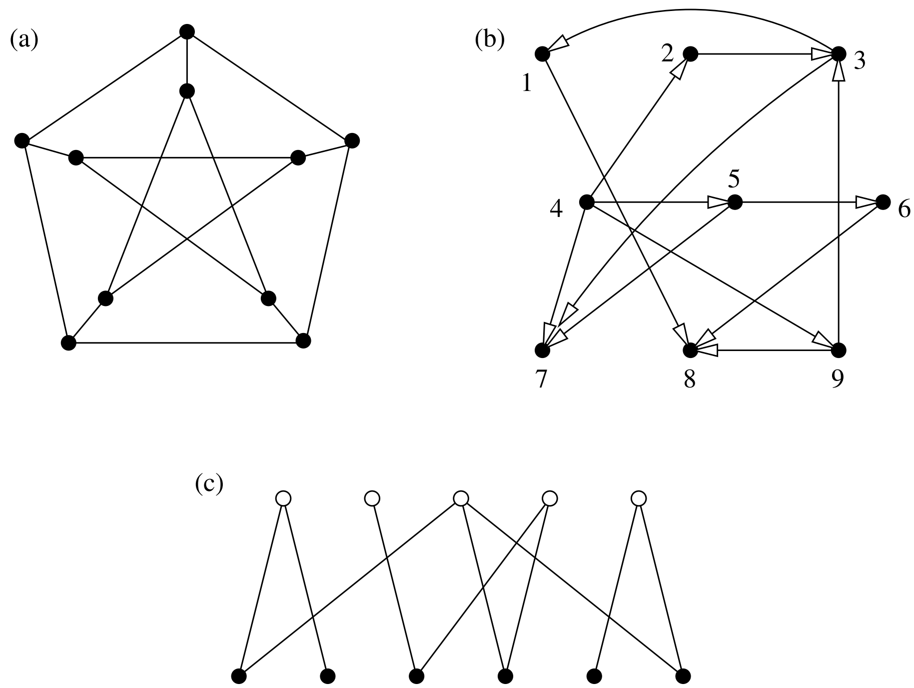
Complex Systems 535/Physics 508: Homework 1

1. **Planar and acyclic graphs:** Here are three small graphs, one undirected, another directed, and the third bipartite:



Draw figures showing the following:

(i) that graph (a) is not planar;

(ii) that graph (b) is not cyclic;

(iii) the bibliographic coupling graph of (b);

(iv) the two one-mode projections of (c).

2. **Eigenvalues of an acyclic digraph:** Consider an acyclic digraph with no self-edges (i.e., no edges connecting vertices to themselves). We showed in class that the adjacency matrix of such a graph can be written in upper triangular form by a suitable labeling of the vertices.

(i) Show that all eigenvalues of the adjacency matrix of such an acyclic digraph are zero.

(ii) Write down an expression for the number of closed loops of length r in a graph in terms of the eigenvalues of the adjacency matrix.

(iii) Hence, or otherwise, show (conversely) that if all eigenvalues of the adjacency matrix are zero the graph must be acyclic.

3. **Lowest eigenvalue of the Laplacian:** Consider a connected graph (i.e., every vertex is reachable from every other) which is undirected with Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$.

   (i) What is the lowest eigenvalue of $\mathbf{L}$ and what is the corresponding eigenvector?

   (ii) How many rows of the Laplacian are linearly independent?

   (iii) Show that an undirected graph is connected if and only if $\lambda_2 > 0$, where λ_2 is the second-smallest eigenvalue of the Laplacian.

   The eigenvalue λ_2 is called the *algebraic connectivity* of the graph, and will come up repeatedly in our study of networks.

4. **Geodesic paths and the adjacency matrix:** Consider the set of all paths from vertex s to vertex t on an undirected graph with adjacency matrix $\mathbf{A}$. Let us give each path a weight equal to α^r, where r is the length of the path.

   (i) Show that the sum of the weights of all the paths from s to t is given by Z_{st}, which is the st element of the matrix $\mathbf{Z} = (\mathbf{I} - \alpha\mathbf{A})^{-1}$. What condition must α satisfy for the sum to converge?

   (ii) Hence, or otherwise, show that the length ℓ_{st} of a geodesic path from s to t, if there is one, is

$$\ell_{st} = \lim_{\alpha \to 0} \frac{\partial \log Z_{st}}{\partial \log \alpha}.$$