Partial derivatives:
\[ \begin{align*}
\frac{d}{dx} f &= \frac{\partial f}{\partial x}, \\
\frac{d}{dy} f &= \frac{\partial f}{\partial y}, \\
\frac{d}{dz} f &= \frac{\partial f}{\partial z}, \\
\frac{\partial x}{\partial y} &= \left[ \frac{\partial y}{\partial x} \right]^{-1}, \\
\frac{\partial x}{\partial z} &= -\frac{\partial x}{\partial y} \frac{\partial y}{\partial z}.
\end{align*} \]

Internal energy:
\[ dU = dQ + dW \] with \( dQ = T \, dS \) and:
- \( dW = -p \, dV \) (fluid pressure/volume system)
- \( dW = f \, dL \) (spring or wire with force \( f \) and length \( L \))
- \( dW = V \, dq \) (capacitor with voltage \( V \) and charge \( q \))
- \( dW = -B \cdot dm \) (magnet with magnetization \( m \) in field \( B \))
- \( dW = \gamma dA \) (surface with surface tension \( \gamma \) and area \( A \))

Thus for example, in a pressure/volume system \( dU = T \, dS - p \, dV \). This applies for irreversible as well as reversible changes, but the individual equalities \( dQ = T \, dS \) and \( dW = -p \, dV \) only apply for reversible ones. Heat capacity at constant \( x \) (where \( x \) is any variable) is in general given by
\[ C_x = T \frac{\partial S}{\partial T} \bigg|_x, \quad \text{e.g.,} \quad C_V = T \frac{\partial S}{\partial T} \bigg|_V \quad \text{and} \quad C_p = T \frac{\partial S}{\partial T} \bigg|_p. \]

Potential functions and Maxwell relations: For pressure volume system
\[ H = U + pV \] (enthalpy), \quad \[ F = U - TS \] (free energy), \quad \[ G = U + pV - TS \] (Gibbs energy).

Similar expressions apply for other types of systems (non-pressure/volume systems). There is one Maxwell relation for each potential function, derived by equating partial second derivatives. For instance
\[ \frac{\partial^2 U}{\partial V \partial S} = \frac{\partial^2 U}{\partial S \partial V} \Rightarrow \frac{\partial T}{\partial V} \bigg|_S = -\frac{\partial p}{\partial S} \bigg|_V. \]

Each heat capacity is the derivative of the corresponding potential function:
\[ C_V = \frac{\partial U}{\partial T} \bigg|_V, \quad C_p = \frac{\partial H}{\partial T} \bigg|_p. \]

Heat engines:

Efficiency of a reversible engine \((T_1 > T_2)\):
\[ \eta = \frac{W}{Q_1} = 1 - \frac{T_2}{T_1}, \quad \eta_R = \frac{Q_2}{W} \] (refrigerator), \quad \[ \eta_H = \frac{Q_1}{W} \] (heat pump).
**Isolated systems:** All microstates equally likely. Most likely macrostate maximizes the Boltzmann entropy $S = k \ln g$, where $g$ is the multiplicity. When non-interacting systems are combined, entropy is additive (i.e., extensive); multiplicity is multiplicative.

**Fixed temperature systems:** States $s$ appear with Boltzmann probability

$$p(s) = \frac{e^{-\varepsilon_s/\tau}}{Z}, \quad Z = \sum_s e^{-\varepsilon_s/\tau},$$

where $\tau = kT$ and $k = 1.38 \times 10^{-23}$ JK$^{-1}$. Macroscopic thermodynamic quantities are then given by

$$U = \tau^2 \left. \frac{\partial \ln Z}{\partial \tau} \right|_V, \quad F = -\tau \ln Z, \quad C = \left. \frac{\partial U}{\partial \tau} \right|_V, \quad \sigma = -\left. \frac{\partial F}{\partial \tau} \right|_V, \quad p = -\left. \frac{\partial F}{\partial V} \right|_\tau.$$

**Sterling’s approximation:** $\ln k! \simeq k \ln k - k$.

**Perfect gas:** Density of states in three dimensions is

$$n(\varepsilon) = \frac{V}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \varepsilon^{1/2},$$

where $I$ is the spin of the particles. ($I = \frac{1}{2}$ for fermions.)

$$Z = \frac{1}{N!} Z_1^N, \quad Z_1 = \frac{V}{(2\pi\hbar^2/m\tau)^{3/2}}, \quad pV = N\tau, \quad \sigma = N\left(\frac{5}{2} - \ln[(2\pi\hbar^2/m\tau)^{3/2}\rho]\right),$$

where $\rho = N/V$ is the number density.

**Photons and phonons:** Density of states is

$$n(\omega) = \frac{V}{\pi^2 c^3} \omega^2 \, d\omega \quad (\text{photons}, \, c \text{ is speed of light}), \quad n(\omega) = \frac{3V}{2\pi^2 v^3} \omega^2 \, d\omega \quad (\text{phonons}, \, v \text{ is speed of sound}).$$

**Systems with variable numbers of particles:** Grand ensemble:

$$\mathcal{Z} = \sum_s e^{-(\varepsilon_s - \mu N_s)/\tau}, \quad U = \tau^2 \left. \frac{\partial \ln \mathcal{Z}}{\partial \tau} \right|_V, \quad \Omega = -\tau \ln \mathcal{Z},$$

$$\sigma = -\left. \frac{\partial \Omega}{\partial \tau} \right|_{V, \mu}, \quad p = -\left. \frac{\partial \Omega}{\partial V} \right|_{\tau, \mu}, \quad N = -\left. \frac{\partial \Omega}{\partial \mu} \right|_{V, \tau}.$$

**Quantum gases:** Number of particles in single-particle state with energy $\varepsilon$ is

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/\tau} + 1} \quad (\text{fermions}), \quad f(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/\tau} - 1} \quad (\text{bosons}).$$