Physics 406: Homework 3

1. **Charging a capacitor:** Returning to the problem of a charging capacitor that we saw earlier in the semester, we can now write a full expression for a change in the internal energy as

$$\mathrm{d}U = T\,\mathrm{d}S + V\,\mathrm{d}q.$$

   (a) Derive a Maxwell relation from this expression for $(\partial T/\partial q)_S$.

   (b) Write down the expressions for small changes in the other three thermodynamic potential functions and the corresponding Maxwell relations. (No need to write out the derivations in full, although you can if it's useful for working them out.)

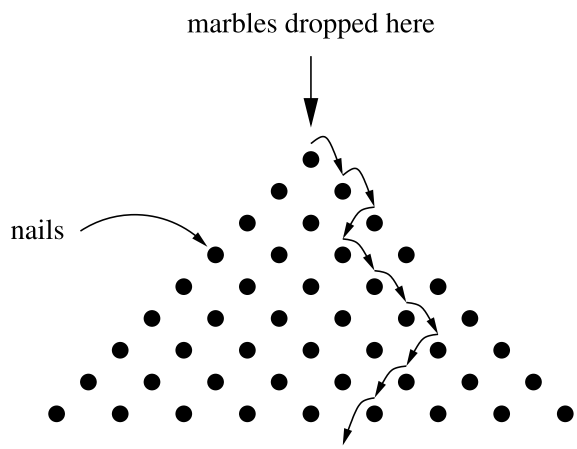
   (c) Suppose we charge the capacitor in thermal isolation. By using the reciprocity rule and one of your Maxwell relations show that

$$\left.\frac{\partial T}{\partial V}\right|_S = -\frac{T}{C_V}\left.\frac{\partial q}{\partial T}\right|_V,$$

   where C_V is the heat capacity at constant voltage. If the capacitance is independent of temperature, what is the change in temperature when we charge the (initially uncharged) capacitor up to one volt?

   (d) Suppose the capacitance varies inversely with temperature $C = a/T$, where a is a constant. Further suppose that the initially uncharged capacitor is at temperature T_1 and that its heat capacity is constant in the temperature range of interest. What then is the final temperature T_2 of the capacitor if we charge it adiabatically up to one volt?

2. **Combinatorics:** A famous carnival toy is a machine that drops a large number of marbles at the top of a pyramid of nails tacked to a board like this:



   The marbles fall down and at each step have a 50% chance of going right or left.

   (a) If there are N rows of nails total, write down an expression for the number of paths $g(N,l,r)$ that a marble can take that make a total of l steps to the left and r steps to the right. Eliminate l and r in favor of the distance $x = r - l$ traveled to the right to get the same number in terms of N and x only. (Note that the distance x is measured horizontally from where the marbles start, i.e., from a line down the middle of the picture above.)

(b) If $N = 10$, how many ways are there of traveling distance $x = 10$ to the right? How many ways are there of traveling distance zero?

(c) If $N = 10$, about how many marbles will have to be dropped before even a single one of them goes all the way to the right-most slot? And how many if $N = 20$?

(d) When many marbles are dropped, what is the expected mean distance $\langle x \rangle$ traveled, averaged over all of them? And what is the standard deviation of the distance?

(e) So if you had to say where a single marble dropped would land, between about which values of x would you feel reasonably confident saying it would end up, if $N = 100$? (If you want to be really precise, you could say which values would you have 90% confidence it would land between, but any sensible answer will do for this question. Saying that x lies between -100 and 100 is not a good answer!)

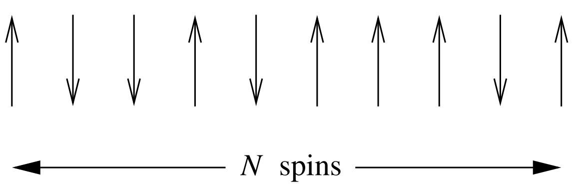
3. **Temperature:** We have seen that for a system of fixed volume and number of particles, the temperature τ and entropy σ are related by

$$\frac{1}{\tau} = \frac{\partial \sigma}{\partial U},$$

where U is the internal energy. Thus we can calculate a rough value for the temperature of a system by comparing a small change ΔU in the internal energy with the corresponding change $\Delta\sigma$ in the entropy thus:

$$\tau = \frac{\Delta U}{\Delta \sigma}. \tag{1}$$

(a) For the system of magnetic "spins" (i.e., dipoles) that we discussed in class:



write down the internal energy U of the system in a magnetic field B (in the direction of the upward-pointing spins) in terms of the magnetic moment m of one of the dipoles, and the spin excess s defined by

$$2s = N_\uparrow - N_\downarrow,$$

where $N_\uparrow$ and $N_\downarrow$ are the numbers of spins pointing up and down respectively. Also write down the multiplicity $g(N, N_\uparrow)$ in terms of s. What is the entropy in terms of s?

(b) What is the change in energy when the spin excess goes from s to $s+1$? Show that the corresponding change in entropy is

$$\Delta\sigma = \ln \frac{\frac{1}{2}N - s}{\frac{1}{2}N + s + 1}.$$

(c) Calculate a rough expression for the temperature τ of the system from Eq. (1) above. What is the value of τ if $N = 20$, $s = 2$, and $m = 1$, $B = 1$?