Douglass Houghton Workshop, 2/14/2022

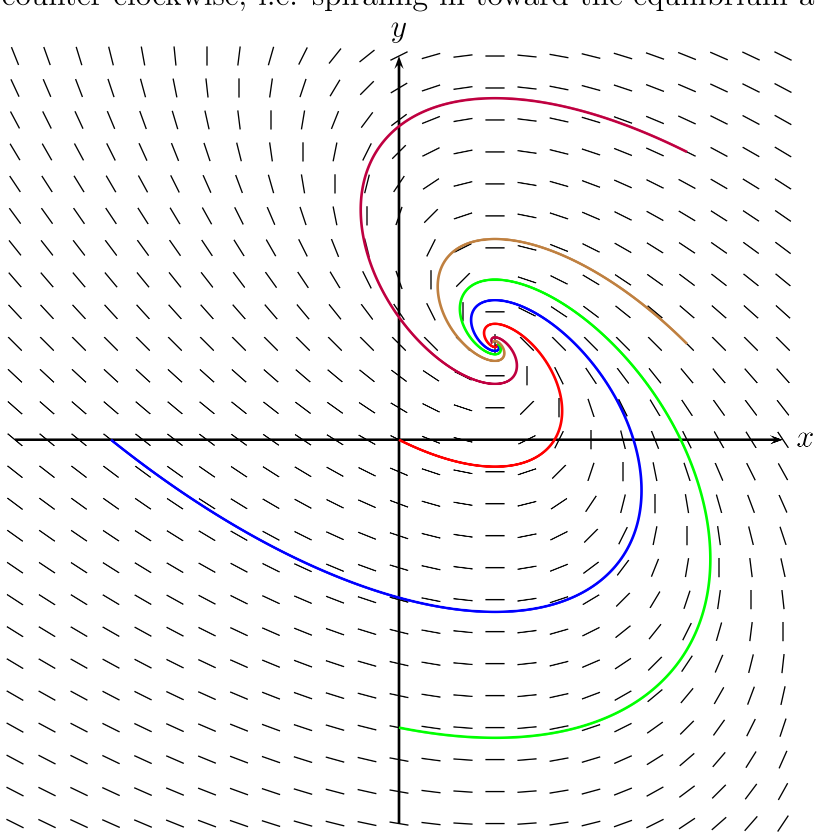
Terren & Marlee

You will recall the tale of Marlee and Terren. It said that if x is Marlee's affection for Terren and y is Terren's affection for Marlee, then their relationship can be modeled by this system of differential equations:

$$dx/dt = 2 - x - y$$
$$dy/dt = x - 1$$

The fact that x is positive in the second equation says that Terren will tend to be more attracted to Marlee when she is attracted to him. The fact that y is negative in the first equation says that the opposite is true for Marlee; she tends to be turned off by Terren's affection. And in fact by her own as well, since the x is negative too. She gets scared when she starts to like him too much.

You all did a great job plotting the slope field for dy/dx on the board, but since it's hard to see the solution curves on a hand-drawn field, I've plotted some of them below with a computer, for $-4 \leq x, y \leq 4$. They are moving generally counter-clockwise around the point $(1, 1)$. You can't tell that by looking, but you can tell it by noting that the differential equations tell us that when x and y are both negative, x is increasing and y is decreasing. So when they're both unhappy with each other, Terren is getting more unhappy but Marlee is getting less unhappy. That means the curves are moving southeast in the third quadrant, so they're going counter-clockwise, i.e. spiraling in toward the equilibrium at $(1, 1)$.



Without the solution curves, it wasn't obvious looking at the slope field whether they are spiraling in or spiraling out. A good way to start to approach the problem might be to shift the variables so that the fixed point moves from $(1, 1)$ to the origin. So let $u = x - 1$ and $v = y - 1$, and if we use Newton's notation of putting a dot over a variable to indicate its derivative with respect to time, we'll have

$$\begin{aligned} \dot{u} &= \dot{x} = 2 - x - y = 2 - (u+1) - (v+1) \Rightarrow \boxed{\dot{u} = -u - v} \\ \dot{v} &= \dot{y} = x - 1 \Rightarrow \boxed{\dot{v} = u}. \end{aligned}$$

That's an improvement; we got the pesky numbers out of there. Now we can tell if the lovers are spiraling in if their distance from the origin is decreasing. That distance is $r = \sqrt{u^2 + v^2}$, and

$$\dot{r} = \frac{1}{2}\left(u^2 + v^2\right)^{-1/2}(2u\dot{u} + 2v\dot{v}) = \frac{1}{r}(u\dot{u} + v\dot{v}) = \frac{1}{r}(u(-u - v) + v(u)) = -\frac{u^2}{r}.$$

Since $\dot{r}$ is clearly negative, r is decreasing, which means they are spiraling in toward the fixed point.

How realistic is this as a model of love? I don't know. But it is certainly true that some people are simple and some are complex, when it comes to matters of the heart. The model is based on a classic predator-prey problem. If x represents the number of rabbits in an ecosystem, for instance, and y the number of wolves, then more rabbits means the wolves are growing, but too many rabbits or too many wolves slows down rabbit growth.

Note that Marlee and Terren can be really unhappy with each other, and still come back to equilibrium eventually. It's very stable: even if something happens to knock them out of equilibrium (say, Terren comes home late one night and they have a fight), they still spiral back in. So you can't always tell what lies in store for a relationship from what you see right now. They might just be circling a whirlpool of love.

In a more complicated system there might be several equilibria, and it would be possible to get knocked from one to another. Some might be stable, inbound spirals, and some might be unstable, outbound spirals. We've all seen friends go through that. Some equilibria might also be just plain attractors or repellors, where you might go straight in or straight out. And some might be saddle points: If you approach from one direction, you go toward the equilibrium, and if you approach from another, you run away from it. Sometimes it takes great care to get to someone's heart.

Love is complicated. Happy Valentine's Day, all. :)