The basic question about improper integrals in Math 116 is whether or not they converge.

**Example 1.** Does \( \int_0^1 \frac{1}{\sqrt{1 - x^2}} \, dx \) converge?

Notice that the function \( \frac{1}{\sqrt{1 - x^2}} \) has a vertical asymptote at \( x = 1 \), so this is an improper integral and we will need to consider the appropriate limit. In this case we can use the fundamental theorem of calculus:

\[
\int_0^1 \frac{1}{\sqrt{1 - x^2}} \, dx = \lim_{b \to 1^-} \int_0^b \frac{1}{\sqrt{1 - x^2}} \, dx = \lim_{b \to 1^-} \left. \arcsin(x) \right|_0^b = \lim_{b \to 1^-} \arcsin(b) - \arcsin(0) = \frac{\pi}{2}.
\]

Therefore the integral converges. \( \square \)

Often we are asked to determine the convergence of an improper integral which is too complicated for us to compute exactly. When this happens we use an integral convergence test.

**Example 2.** Does \( \int_2^\infty \frac{x^2 + x + 1}{x^3 - \sqrt{x}} \, dx \) converge?

Since it looks difficult to find an antiderivative of \( \frac{x^2 + x + 1}{x^3 - \sqrt{x}} \), we need a different approach. For large values of \( x \) we have

\[
\frac{x^2 + x + 1}{x^3 - \sqrt{x}} \approx \frac{x^2}{x^3} = \frac{1}{x},
\]

so we should expect this integral to diverge since \( \int_2^\infty \frac{1}{x} \, dx \) diverges by the \( p \)-test. To show divergence we use the (direct) comparison test. (This is referred to in §7.7 of our textbook as "The Comparison Test".) Note that the inequality

\[
\frac{x^2 + x + 1}{x^3 - \sqrt{x}} \geq \frac{x^2}{x^3} = \frac{1}{x}
\]

holds for all \( x \) in \([2, \infty)\). The integral \( \int_2^\infty \frac{1}{x} \, dx \) diverges by the \( p \)-test with \( p = 1 \), so \( \int_2^\infty \frac{x^2 + x + 1}{x^3 - \sqrt{x}} \, dx \) diverges by direct comparison. \( \square \)

The direct comparison test works in many cases, but there are situations where the simplest comparison we would like to make is not valid.

**Example 3.** Does \( \int_2^\infty \frac{x^2 - x - 1}{x^3 + \sqrt{x}} \, dx \) converge?

This integral looks almost identical to the one in the previous example, but some of the signs have been flipped. For large values of \( x \) we still have

\[
\frac{x^2 - x - 1}{x^3 + \sqrt{x}} \approx \frac{x^2}{x^3} = \frac{1}{x},
\]
but we now get the inequality
\[ \frac{x^2 - x - 1}{x^3 + \sqrt[3]{x}} \leq \frac{1}{x} \]
for all \( x \) in \([2, \infty)\). This does not help us show that the original improper integral diverges because of the direction of the inequality.

We can, however, find that \( \frac{1}{4}x^2 \leq x^2 - x - 1 \) and \( 2x^3 \geq x^3 + \sqrt[3]{x} \) for all \( x \) in \([2, \infty)\). Combining these, we find
\[ \frac{1}{8x} \leq \frac{x^2 - x - 1}{x^3 + x^{1/3}} \]
for \( x \) in \([2, \infty)\). This lets us use the comparison test, but it took more work to come up with an appropriate function to compare with. \( \square \)

The limit comparison test gives us another strategy for situations like Example 3.

**Limit comparison test (LCT) for improper integrals:** Suppose \( f(x) \) and \( g(x) \) are positive, continuous functions defined on \((a, \infty)\) such that
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = c \]
where \( c \) is a positive number. (In particular, \( c \neq 0 \) and \( c \neq \infty \).) Then \( \int_a^\infty f(x) \, dx \) and \( \int_a^\infty g(x) \, dx \) either both converge or both diverge.

Let’s return to Example 3 to see how the LCT works. We want to show that \( \int_2^\infty \frac{x^2 - x - 1}{x^3 + x^{1/3}} \, dx \) diverges. Using the notation from the LCT, we let
\[ f(x) = \frac{x^2 - x - 1}{x^3 + x^{1/3}} \]
and choose \( g(x) = \frac{1}{x} \), which was the function we originally hoped to compare \( f(x) \) with. You can check that both of these functions are positive and continuous on \([2, \infty)\). Next we compute
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^2 - x - 1}{x^3 + x^{1/3}} = \frac{1}{\frac{1}{x}} = \lim_{x \to \infty} \frac{x^3 - x^2 - x}{x^3 + x^{1/3}} = 1. \]
Since the limit is 1, which is not 0 or \( \infty \), we can apply the LCT. The integral \( \int_2^\infty \frac{1}{x} \, dx \) diverges by the \( p \)-test with \( p = 1 \); thus the LCT tells us that \( \int_2^\infty f(x) \, dx \) must also diverge.

The limit comparison test is used when we want to determine whether an improper integral \( \int_a^\infty f(x) \, dx \) converges, but the function \( f(x) \) is too complicated for us to either compute an antiderivative or to easily find a comparison with a simpler function. To choose the function \( g(x) \) we look for something that is simple and that grows at a rate that is similar to \( f(x) \). The limit
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = c \]
tells us how the growth rates of \( f(x) \) and \( g(x) \) compare for large values of \( x \).

1. If \( c = 0 \), this means that \( g(x) \) grows much faster than \( f(x) \).
2. If \( c = \infty \), this means that \( f(x) \) grows much faster than \( g(x) \).
3. If \( c \neq 0 \) or \( \infty \), this means that \( f(x) \) and \( g(x) \) grow at a similar rate.

Note that if this limit does not exist, is 0 or \( \infty \), then the LCT is inconclusive, and cannot be applied.
**Steps for using the LCT:** Use the LCT when trying to determine whether \( \int_{a}^{\infty} f(x) \, dx \) converges and the function \( f(x) \) is positive and looks complicated.

1. Find a function \( g(x) \) which grows at a similar rate as \( f(x) \) and is simpler.
2. Compute
   \[ c = \lim_{x \to \infty} \frac{f(x)}{g(x)}. \]
   Make sure the limit exists and is not 0 or \( \infty \).
3. Check by some other means whether \( \int_{a}^{\infty} g(x) \, dx \) converges or diverges.
4. Cite the LCT to conclude whether \( \int_{a}^{\infty} f(x) \, dx \) converges or diverges.

Following are some additional examples in which the LCT is used.

**Example 4.** Determine whether \( \int_{1}^{\infty} \frac{7\sqrt{x} + \sqrt[3]{x}}{\sqrt{x^5 + \cos(x)}} \, dx \) converges.

The function \( f(x) = \frac{7\sqrt{x} + \sqrt[3]{x}}{\sqrt{x^5 + \cos(x)}} \) looks pretty complicated. In this situation we turn immediately to the limit comparison test. What function do we choose for \( g(x) \)? Looking at the most significant terms in the numerator and denominator we see that for large values of \( x \),

\[ \frac{7\sqrt{x} + \sqrt[3]{x}}{\sqrt{x^5 + \cos(x)}} \approx \frac{\sqrt{x}}{\sqrt{x^5}} = \frac{1}{x^2}. \]

Let \( g(x) = \frac{1}{x^2} \). Next compute the necessary limit:

\[ \lim_{x \to \infty} \frac{7\sqrt{x} + \sqrt[3]{x}}{\sqrt{x^5 + \cos(x)}} = \lim_{x \to \infty} \frac{7x^{5/2} + x^{7/3}}{x^{5/2} + \cos(x)}. \]

In the numerator, \( 7x^{5/2} \) dominates \( x^{7/3} \), that is, when \( x \) is very large, \( 7x^{5/2} \) becomes so much larger than \( x^{7/3} \) that the latter term becomes negligible. In the denominator, \( \cos(x) \) is between \(-1 \) and \( 1 \), and so is dominated by \( x^{5/2} \) when \( x \) is very large. This tells us that

\[ \lim_{x \to \infty} \frac{7x^{5/2} + x^{7/3}}{x^{5/2} + \cos(x)} = 7. \]

Since the limit is 7 and not 0 or \( \infty \), the LCT applies. The integral \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) converges by the \( p \)-test with \( p = 2 \). Thus, the LCT tells us that

\[ \int_{1}^{\infty} \frac{7\sqrt{x} + \sqrt[3]{x}}{\sqrt{x^5 + \cos(x)}} \, dx \]

must also converge. \( \square \)

**Example 5.** Does \( \int_{0}^{\infty} \frac{e^{x} + x^{3}}{e^{3x} - x^{3}} \, dx \) converge?

For large \( x \),

\[ \frac{e^{x} + x^{3}}{e^{3x} - x^{3}} \approx \frac{e^{x}}{e^{3x}} = e^{-2x}. \]

Let \( g(x) = e^{-2x} \). Then we compute

\[ \lim_{x \to \infty} \frac{e^{x} + x^{3}}{e^{-2x}} = \lim_{x \to \infty} \frac{e^{2x}(e^{x} + x^{3})}{e^{3x} - x^{3}}. \]

In the numerator, \( e^{x} \) dominates \( x^{3} \) when \( x \) is large, which means numerator overall is dominated by \( e^{3x} \). In the denominator, \( e^{3x} \) dominates \( x^{3} \) when \( x \) is large. Thus this limit is 1.
Since the limit converged to 1 and not 0 or $\infty$, the LCT applies. The integral $\int_0^\infty e^{-2x} \, dx$ converges by the exponential decay test, and so it follows that

$$\int_0^\infty \frac{e^x + x^3}{e^{3x} - x^3} \, dx$$

covers by the LCT. □

Our next example highlights one pitfall of the LCT.

**Example 6.** Determine whether $\int_1^\infty \frac{x(3 + \cos(x))}{x^3 - e^{-x}} \, dx$ converges.

A rough estimate tells us that for large $x$,

$$\frac{x(3 + \cos(x))}{x^3 - e^{-x}} \approx \frac{x}{x^3} = \frac{1}{x^2}.$$

But we run into trouble when we try to compute the required limit for the LCT:

$$\lim_{x \to \infty} \frac{x(3 + \cos(x))}{x^3 - e^{-x}} = \lim_{x \to \infty} \frac{x^3(3 + \cos(x))}{x^3 - e^{-x}} = \lim_{x \to \infty} \frac{3 + \cos(x)}{1 - \frac{e^{-x}}{x^3}}.$$

We get to the last equality by dividing the numerator and denominator by $x^3$. Here we can see that the denominator will approach 1 as $x$ approaches infinity, but the numerator will never approach a single number, as $\cos(x)$ never stops oscillating through values in $[-1, 1]$. To deal with this we use the inequality

$$3 + \cos(x) \leq 4$$

to see that

$$\frac{x(3 + \cos(x))}{x^3 - e^{-x}} \leq \frac{4x}{x^3 - e^{-x}}.$$

Now we wish to apply the LCT with our new function $f(x) = \frac{4x}{x^3 - e^{-x}}$ and $g(x) = \frac{1}{x^2}$. We find

$$\lim_{x \to \infty} \frac{4x}{x^3 - e^{-x}} = \lim_{x \to \infty} \frac{4x^3}{x^3 - e^{-x}} = \lim_{x \to \infty} \frac{4}{1 - \frac{e^{-x}}{x^3}} = 4.$$

Since $\int_1^\infty \frac{1}{x^2}$ converges by the $p$-test with $p = 2$, we apply the LCT as we had hoped to show that

$$\int_1^\infty \frac{4x}{x^3 - e^{-x}} \, dx$$

covers, and then use the inequality

$$\frac{x(3 + \cos(x))}{x^3 - e^{-x}} \leq \frac{4x}{x^3 - e^{-x}}$$

to conclude that our original integral converges by the direct comparison test. □