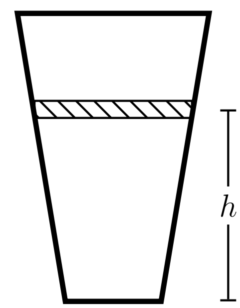
Douglass Houghton Workshop, Section 2, Thu 03/19/20

Worksheet Never Gonna Give You Up

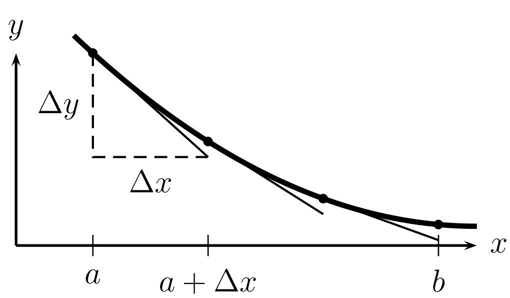
1. Find the area of the finite region that is bounded by the y-axis, the line $y = 1$, and the graph of $y = x^{1/4}$ in two ways:
   (a) By integrating with respect to x and
   (b) By writing x as a function of y and integrating with respect to y.

2. Suppose a Solo cup has radii R_1 cm and R_2 cm and height H cm.



   (a) Consider a disk-shaped slice of the cup which is a height h above the bottom. What is its radius, in terms of h? Hint: The sides of the cup are straight, so the radius is a linear function of h.
   (b) If the thickness of the disk is Δh, what is its volume?
   (c) Find the volume of the cup, and simplyfy as much as possible.
   (d) Simplify your formula in two special cases: where $R_1 = 0$ and where $R_1 = R_2$.

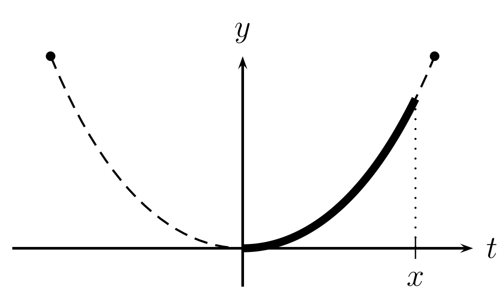
3. How can we compute the length of a curve $y = f(x)$? Consider cutting it up into small pieces, and approximating each piece with a line segment, as in the picture below.



   (a) How long is the first piece? It is tangent to the curve at a.
   (b) How long is the ith piece?
   (c) Write the left-hand Riemann sum for the length of the curve from a to b.
   (d) Now make it into an integral, which will be our formula for arc length.

4. In our quest to determine the shape of a hanging chain, we have found that the forces on a portion of the chain obey a certain relationship: if $m(x)$ is the mass of the chain between the middle and position x, T_0 is the tension in the chain at the bottom, and $y = F(x)$ is the shape of the chain, then in order to make the forces balance we must have:

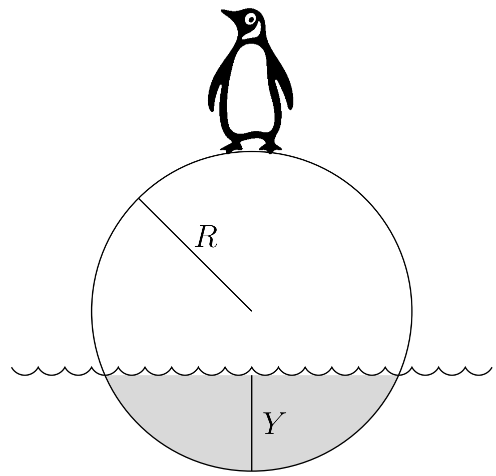
$$\frac{m(x)g}{T_0} = F'(x).$$



   (a) How could you calculate $m(x)$ if you knew $F(x)$?

(b) Some of that we know how to do. Use it to modify the equation above. Feel free to combine unknown constants into one, and simplify as much as possible.

5. The buoyancy force on a floating object is proportional to the volume of water it displaces. Using this fact, whimsical ecologists plan to study the weights of penguins by floating beachballs on the ocean and enticing penguins to climb on top. They then measure the depth the ball sinks to, and thereby deduce the penguin's weight.

   So given a beachball of radius R that is partially submerged in the water, find a formula for the volume of the ball which is below the water line when its bottom is at depth Y. Check that your formula makes sense for the values $Y = 0$, $Y = R$, and $Y = 2R$.

6. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1x + a_2x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

   where, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

   Suppose $f(x) = F_0 + F_1x + F_2x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

   (a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

   (b) What happens when you add those two together? Compare with $f(x)/x$.

   (c) Deduce a simple formula for $f(x)$.