Douglass Houghton Workshop, Section 2, Thu 02/27/20

Worksheet Karma

With a lot of hard work, we filled the table to the right with the values of $\int_{-\pi}^{\pi} f(x)g(x)\,dx$, where f is the row and g is the column, and m and n are positive integers.

	1	$\sin(nx)$	$\cos(nx)$
1	2π	0	0
$\sin(mx)$	0	$\begin{cases} \pi & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$	0
$\cos(mx)$	0	0	$\begin{cases} \pi & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$

The implication was that for a function of the form

$$\begin{aligned} f(x) &= a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) \; + \; \cdots \\ &\quad + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) \; + \; \cdots \end{aligned} \tag{1}$$

integrating against a sine or cosine function makes almost all the terms 0, so for $n \geq 1$:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

Writing a function in the form of Equation (1) is called finding the **Fourier Series** of the function.

1. Let's compute the Fourier series for $f(x) = x^2$.
   (a) Compute a_0.
   (b) Fill in the table to the right.
   (c) Find the a_n and b_n for $f(x) = x^2$.

n	1	2	3	4	5	6
$\sin(n\pi)$						
$\sin(-n\pi)$						
$\cos(n\pi)$						
$\cos(-n\pi)$						

2. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.
   (a) $e^x =$
   (b) $e^{-x} =$
   (c) $\cos(x) =$
   (d) $\sin(x) =$
   (e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$
   (f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

3. (Fall, 2014) Prove whether these series converge or diverge:
   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$

4. (Fall, 2007) Find the interval of convergence of $\displaystyle\sum_{n=3}^{\infty} \frac{(3-x)^{3n}}{8^n(n-2)}$.