Douglass Houghton Workshop, Section 2, Thu 02/20/20

Worksheet If Music Be the Food of Love, Play On

With a lot of hard work, we filled the table to the right with the values of $\int_{-\pi}^{\pi} f(x)g(x)\,dx$, where f is the row and g is the column, and m and n are positive integers.

	1	$\sin(nx)$	$\cos(nx)$
1	2π	0	0
$\sin(mx)$	0	$\begin{cases} \pi & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$	0
$\cos(mx)$	0	0	$\begin{cases} \pi & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$

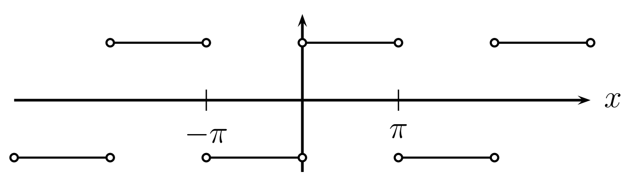
The implication was that for a function of the form

$$f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots \\ + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots \tag{1}$$

integrating against a sine or cosine function makes almost all the terms 0, so for $n \geq 1$:

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0, \quad \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \pi a_n, \quad \text{and} \quad \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \pi b_n.$$

1. Suppose that you have a function with the form of Equation (1), but you don't know the coefficients. You can, however, find integrals like the ones above.
   (a) How can you find a_1, the coefficient of $\cos(x)$?
   (b) How can you find a_n and b_n?
2. Consider the square wave:
   $$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } \quad 0 < x < \pi \end{cases}$$
   and that pattern is repeated every 2π.



   Suppose the square wave can be written in terms of sines and cosines, as in Equation (1) above. Find the a_n and the b_n.
3. (This problem appeared on a Winter, 2009 Math 116 exam) The quantity
   $$\int_1^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$$
   roughly models the resistance that football-shaped plankton encounter when falling through water. Note that $a = 1$, $b = 2$, and $c = 3$ are constants that describe the dimensions of the plankton. Find a value of M for which
   $$\int_1^M \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$$
   differs from the original model of resistance by at most 0.001. Hint: make use of the integral and the comparison test.

4. Last time it appeared we showed that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$. Crazy!

   (a) We got that series by considering derivatives and plugging in $x = 0$. See if you can deduce a series for $\cos(x)$ the same way, by starting with

   $$\cos(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

   (b) Test it at $x = \pi$, by adding up all the terms through x^{10}. Is it close to what you expect it to be?

   (c) Do the same for $\sin(x)$.

   (d) Now systemetize the result: if we have a function $f(x)$ which has derivatives, how do we find its series? Find formulas for a_0, a_1, etc. in terms of f.

5. (From the Winter, 2011 Math 116 final exam) Determine whether the following series converge or diverge, and justify your answer using one or more convergence tests. No credit without justification.

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{\sqrt{n^5 + n^2}}$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n(1+\ln(n))}$

6. (From the Winter, 2010 Math 116 Final Exam) Discover whether the following series converge or diverge, and justify using an appropriate converge test or tests. No credit without justification.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}}{n^2-1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{\sin(n)}{n^2-3}$

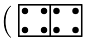
7. In the course of our probability investigations we learned that if $-1 < t < 1$ then

   $$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \cdots$$

   where the dots at the end mean the sum goes on forever.

   (a) Take the derivative of both sides, and see what you get.

   (b) Suppose among a certain group of people, 54% get 1 scoop of ice cream, 32% get 2 scoops, and 14% get 3 scoops. What is the average number of scoops per person?

   (c) One question that a casino might want to answer about a game is how long it is likely to go on. Consider the hard-eight () bet in craps, for instance, where you win with probability $W = \frac{1}{36}$, lose with probability $L = \frac{10}{36}$, and keep rolling with probability $C = \frac{25}{36}$. What is the probability that the bet is resolved on the first roll? The second roll? The kth roll?

   (d) What is the average number of rolls before the bet is resolved?