Douglass Houghton Workshop, Section 2, Thu 01/23/20

Worksheet Do the Right Thing

1. Consider the **gamma function**: $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$, for $x > 0$.

   (a) Use integration by parts to prove that $\Gamma(x+1) = x\Gamma(x)$.

   (b) Show that $\Gamma(1) = 1$. Then fill in this chart, using part (a):

x	1	2	3	4	5	6
$\Gamma(x)$						

   (c) So if x is a positive integer, what is $\Gamma(x)$?

2. Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx$ where m and n are positive integers. (You might want to graph a few examples.)

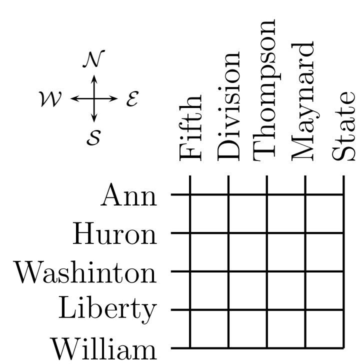
3. (Fall, 2007) For this problem, $\int_1^5 g(x)\,dx = 12$ and $f(x) = 2x - 9$. Some values of $g(x)$ are:

x	1	2	3	4	5
$g(x)$	0.1	1.5	2	5	10

   (a) Find $\int_5^7 g(f(x))\,dx$. (b) Find $\int_1^5 f(x)g'(x)\,dx$.

   (c) Find $\int_1^5 \frac{g'(x)}{g(x)(g(x)+1)}\,dx$.

4. A small section of downtown Ann Arbor is shown to the right. Copy the map onto the board.



   (a) Suppose Myrka lives at the corner of Washington and Thompson, and she needs to get to class at Mason Hall, which is at State and William. She doesn't want to walk out of her way, so she will only go east and south. Still, she has some choices. How many ways are there to get to class?

   (b) Interesting, I wonder what that number means? Write your answer to part (a) at the corner of Washington and Thompson. Now pick a different starting corner, and figure out how many ways there are to get to class from there. Repeat, writing your answers on the board at the relevant corner.

   (c) What's the pattern?

   (d) Explain why the pattern must continue to hold, no matter how big the city is.

5. We have a new tool for evaluating limits, called *L'Hôpital's Rule*. It allows us to resolve some limits we couldn't do last fall. In particular, we can prove our favorite result, about Michael Phelps's towel.

   (a) Recall our model of toweling: water spreads out evenly over Michael and the towel. So if Michael has area $1\,\mathrm{m}^2$ and starts with wetness W, after using a towel of size T his new wetness will be
   $$W\left(\frac{1}{1+T}\right).$$
   Suppose instead he divides his towel of size T into n parts. How wet will he be if he starts with 1 liter of water on him?

   (b) What happens if he divides the towel into more and more pieces? Let
   $$L = \lim_{n\to\infty}(\text{the formula you found in part (5a)}).$$
   Take the ln of both sides of the equation above. It's OK to move the ln inside the limit, because ln is a continuous function.

   (c) Let $p = 1/n$. As $n \to \infty$, $p \to 0$. So rewrite your limit with p's instead of n's.

   (d) In order to use L'Hôpital's Rule, you need to have something of the form $0/0$ or ∞/∞. So get you limit in that form, and resolve it.

6. A while back we showed that if a parabola $y = p(x)$ goes through the three points $(-1, R)$, $(0, S)$, and $(1, T)$, then $\int_{-1}^{1} p(x)\,dx = \frac{1}{3}(R + 4S + T)$. Suppose now that a parabola $y = q(x)$ goes through the points (x_0, y_0), (x_1, y_1), and (x_2, y_2), with $x_0 = x_1 - h$ and $x_2 = x_1 + h$. (So x_1 is midway between the other x's.)

   (a) Find a linear substitution that makes $\int_{x_0}^{x_2} q(x)\,dx$ into an integral from -1 to 1. Is the integrand still a parabola?

   (b) Use our previous result to calculate the area under $q(x)$ from x_0 to x_2.

7. Currently 95% of Michigan kindergarteners have been vaccinated for measles. The measles vaccine is 93% effective, meaning that 7% of vaccinated children who are exposed to the disease will contract it, and the rest will not. That contrasts with a 10% immunity among unvaccinated children.

   (a) Suppose that all children in the community are exposed to the measles vaccine, and fill in the following table of possibilities. For instance, the upper-left corner is the probability that a randomly-chosen child is vaccinated *and* contracts measles.

		Vaccinated?	
		Yes	No
Gets measles?	Yes		
	No		

   (b) What proportion of the students who contract measles were vaccinated?

   (c) What does that mean about whether you should vaccinate your child?