Douglass Houghton Workshop, Section 1, Wed 02/26/20

Worksheet If You Tickle Us, Do We Not Laugh?

With a lot of hard work, we filled the table to the right with the values of $\int_{-\pi}^{\pi} f(x)g(x)\,dx$, where f is the row and g is the column, and m and n are positive integers.

	1	$\sin(nx)$	$\cos(nx)$
1	2π	0	0
$\sin(mx)$	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$	0
$\cos(mx)$	0	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$

The implication was that for a function of the form

$$\begin{aligned} f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) \;+\; \cdots \\ +\, b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) \;+\; \cdots \end{aligned} \tag{1}$$

integrating against a sine or cosine function makes almost all the terms 0, so for $n \geq 1$:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

Writing a function in the form of Equation (1) is called finding the **Fourier Series** of the function.

1. Let's compute the Fourier series for $f(x) = x^2$.
   (a) Compute a_0.
   (b) Fill in the table to the right.
   (c) Find the a_n and b_n for $f(x) = x^2$.

n	1	2	3	4	5	6
$\sin(n\pi)$						
$\sin(-n\pi)$						
$\cos(n\pi)$						
$\cos(-n\pi)$						

2. Find the probability of winning the pass bet in craps.

3. Last time it appeared we showed that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$. Crazy!
   (a) We got that series by considering derivatives and plugging in $x = 0$. See if you can deduce a series for $\cos(x)$ the same way, by starting with
   $$\cos(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
   (b) Test it at $x = \pi$, by adding up all the terms through x^{10}. Is it close to what you expect it to be?
   (c) Do the same for $\sin(x)$.
   (d) Now systemetize the result: if we have a function $f(x)$ which has derivatives, how do we find its series? Find formulas for a_0, a_1, etc. in terms of f.