Douglass Houghton Workshop, Section 1, Mon 02/24/20

Worksheet How far that little candle throws its beams!

So shines a good deed in a naughty world.

With a lot of hard work, we filled the table to the right with the values of $\int_{-\pi}^{\pi} f(x)g(x)\,dx$, where f is the row and g is the column, and m and n are positive integers.

	1	$\sin(nx)$	$\cos(nx)$
1	2π	0	0
$\sin(mx)$	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$	0
$\cos(mx)$	0	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$

The implication was that for a function of the form

$$\begin{aligned} f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) \, + \, \cdots \\ + \, b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) \, + \, \cdots \end{aligned} \tag{1}$$

integrating against a sine or cosine function makes almost all the terms 0, so for $n \geq 1$:

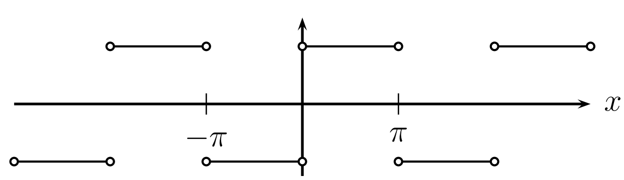
$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0, \quad \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \pi a_n, \quad \text{and} \quad \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \pi b_n.$$

1. Generalize: Suppose that you have a function with the form of Equation (1), but you don't know the coefficients. You can, however, find integrals like the ones above.
   - How can you find a_1, the coefficient of $\cos(x)$?
   - How can you find a_n and b_n?

2. Consider the square wave:

   $$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } \quad 0 < x < \pi \end{cases}$$

   and that pattern is repeated every 2π.

   Suppose the square wave can be written in terms of sines and cosines, as in Equation (1) above. Find the a_n and the b_n.

3. Find the probability of winning the pass bet in craps.

4. (From the Winter, 2011 Math 116 final exam) Determine whether the following series converge or diverge, and justify your answer using one or more convergence tests. No credit without justification.

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n^2}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{3n-2}{\sqrt{n^5+n^2}}$ (c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n(1+\ln(n))}$

5. (From the Winter, 2010 Math 116 Final Exam) Discover whether the following series converge or diverge, and justify using an appropriate converge test or tests. No credit without justification.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}}{n^2-1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{\sin(n)}{n^2-3}$

6. Last time it appeared we showed that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$. Crazy!

   (a) We got that series by considering derivatives and plugging in $x = 0$. See if you can deduce a series for $\cos(x)$ the same way, by starting with

$$\cos(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

   (b) Test it at $x = \pi$, by adding up all the terms through x^{10}. Is it close to what you expect it to be?

   (c) Do the same for $\sin(x)$.

   (d) Now systemetize the result: if we have a function $f(x)$ which has derivatives, how do we find its series? Find formulas for a_0, a_1, etc. in terms of f.

7. (Adapted from a Fall, 2003 Math 116 Exam)

   (a) Express the number

$$.135135135\overline{135}$$

   as the sum of a geometric series.

   (b) Use the infinite geometric series formula to express that same number as a fraction in lowest terms.

8. Between 1980 and 2009, there were 7015 Division I basketball games which were tied at the end of regulation play, resulting in at least one overtime period. Suppose p is the probability that an overtime period ends in a tie.

   (a) How many 2nd overtimes do you expect there were? How many 3rd overtimes? How many kth overtimes?

   (b) There were in fact 8568 total overtime periods between 1980 and 2009. Use that to estimate p.

   (c) About how many games went 6 or more overtimes, do you guess?