Douglass Houghton Workshop, Section 1, Mon 01/27/20

Worksheet Damn the Torpedoes

1. Consider the **gamma function**: $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$, for $x > 0$.

   (a) Use integration by parts to prove that $\Gamma(x+1) = x\Gamma(x)$.

   (b) Show that $\Gamma(1) = 1$. Then fill in this chart, using part (a):

x	1	2	3	4	5	6
$\Gamma(x)$						

   (c) So if x is a positive integer, what is $\Gamma(x)$?

2. (Fall, 2007) For this problem, $\int_1^5 g(x)\,dx = 12$ and $f(x) = 2x - 9$. Some values of $g(x)$ are:

x	1	2	3	4	5
$g(x)$	0.1	1.5	2	5	10

   (a) Find $\int_5^7 g(f(x))\,dx$. (b) Find $\int_1^5 f(x)g'(x)\,dx$.

   (c) Find $\int_1^5 \frac{g'(x)}{g(x)(g(x)+1)}\,dx$.

3. We have a new tool for evaluating limits, called *L'Hôpital's Rule*. It allows us to resolve some limits we couldn't do last fall. In particular, we can prove our favorite result, about Michael Phelps's towel.

   (a) Recall our model of toweling: water spreads out evenly over Michael and the towel. So if Michael has area $1\,\mathrm{m}^2$ and starts with wetness W, after using a towel of size T his new wetness will be

$$W\left(\frac{1}{1+T}\right).$$

   Suppose instead he divides his towel of size T into n parts. How wet will he be if he starts with 1 liter of water on him?

   (b) What happens if he divides the towel into more and more pieces? Let

$$L = \lim_{n\to\infty}(\text{the formula you found in part (4a)}).$$

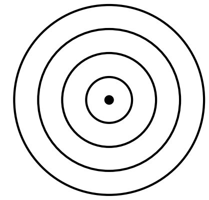
   Take the ln of both sides of the equation above. It's OK to move the ln inside the limit, because ln is a continuous function.

   (c) Let $p = 1/n$. As $n \to \infty$, $p \to 0$. So rewrite your limit with p's instead of n's.

   (d) In order to use L'Hôpital's Rule, you need to have something of the form $0/0$ or ∞/∞. So get you limit in that form, and resolve it.

4. Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx$ where m and n are positive integers. (You might want to graph a few examples.)

5. Consider a game of "continuous darts". The board is circular, as you expect, with radius 1. The goal is to get as close to the middle as possible. If a dart lands a distance r from the bullseye, its score is $1-r$. (So every number between 0 and 1 is a possible score.)



A novice player throws a dart which lands randomly somewhere on the board. That means that for any region R on the board,

$$\text{Prob(dart lands in } R) = \frac{\text{area of } R}{\text{area of board}}.$$

(a) Fill in the table with the probabilities that the dart scores **below** the given value.

x	0	1/4	1/2	3/4	1
Prob(score $< x$)					

(b) Let x be any number. Find the probability that the score is less than x.

(c) Find the median score.

6. The function you found in (5b) above is called the **cumulative distribution function** or **CDF** of the score. Let's call it $P(x)$.

(a) Use $P(x)$ to find the probability that the score is between 1/3 and 2/3.

(b) How would you use $P(x)$ to find the probability that a score is between a and b?

(c) Hmmm, that answer reminds you of the First Fundamental Theorem of Calculus, I bet. Can you write it as an integral?

The derivative of $P(x)$ is called the **probability density function** or **PDF** of the score. Let's call if $p(x)$.

7. Currently 95% of Michigan kindergarteners have been vaccinated for measles. The measles vaccine is 93% effective, meaning that 7% of vaccinated children who are exposed to the disease will contract it, and the rest will not. That contrasts with a 10% immunity among unvaccinated children.

(a) Suppose that all children in the community are exposed to the measles vaccine, and fill in the following table of possibilities. For instance, the upper-left corner is the probability that a randomly-chosen child is vaccinated *and* contracts measles.

		Vaccinated?	
		Yes	No
Gets measles?	Yes		
	No		

(b) What proportion of the students who contract measles were vaccinated?

(c) What does that mean about whether you should vaccinate your child?