Douglass Houghton Workshop, Section 2, Thu 04/04/19

Worksheet One Ring to Rule Them All

1. There is nothing special at latitude 14°38′53″ N, longitude 78°6′28″ W. It's just a point in the ocean. But, if you were to shoot a neutrino from the middle of the Diag (latitude 42°16′36″ N, longitude 83°44′15″ W) to that point, through the earth's crust, its deepest point would be directly under a very interesting place. Find that place.

2. Last time we found a remarkable power series:

$$\frac{x}{1-x-x^2} = x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+21x^8+34x^9+55x^{10}+\cdots = \sum_{n=0}^{\infty} F_n x^n$$

where F_n is the nth Fibonacci number, defined by $F_n = F_{n-1} + F_{n-2}$. That's already a pretty cool result. But if we play our cards right, we can parlay it into a non-recursive formula for the Fibonacci numbers.

(a) If a is a constant, what is the power series for $\frac{1}{1-ax}$ about $x = 0$?

(b) Verify that if $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ then $(1-\alpha x)(1-\beta x) = 1-x-x^2$.

(c) Now suppose we could split the generating function above like this:

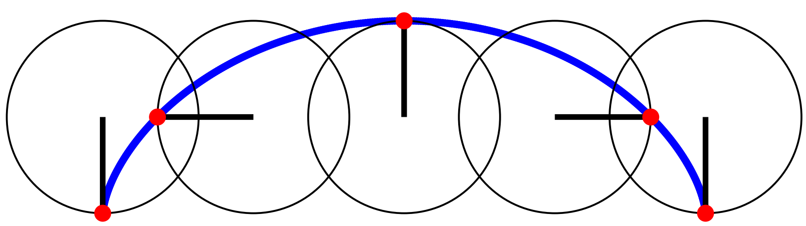
$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

for some constants A and B. Find what A and B must be to make the equation above work for all values of x.

(d) Now find the series for $\frac{A}{1-\alpha x}$ and $\frac{B}{1-\beta x}$, in Σ form, and add them together to get a formula for the Fibonacci numbers.

3. Find the probability of winning the pass bet in craps.

4. Suppose you are watching Tahmina ride a unicycle in the dark. The only thing you can see is a red reflector which is on the outer edge of the front wheel. As Tahmina moves from left to right, you see the reflector trace out the path below:



(a) The wheels in the picture represent snapshots taken every quarter second. The dots are the reflector. Assuming the wheel's center in the first snapshot is $(0, 0)$ and the radius of the wheel is r, fill in the table below with the center's position and the reflector's position at time t.

t	0 sec	1/4 sec	1/2 sec	3/4 sec	1 sec
x_c = center's x	0				
y_c = center's y	0				
x_r = reflector's x					
y_r = reflector's y					

(b) Find formulas for x_c, y_c, x_r, and y_r in terms of t.

(c) Find the exact distance traveled by the reflector in one minute. No approximations!

5. We found an amazing formula last time:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

where $i = \sqrt{-1}$. We got it by using Taylor series for our favorite functions, and plugging in $i\theta$ to see what happens.

The set of *complex numbers* is the set of all things of the form $x + iy$, where x and y are real numbers. You know that real numbers can be represented on a (one-dimensional) number line. Similarly, complex numbers can be represented as points in a (two-dimensional) plane. So $x + iy$ is plotted as the point (x, y).

(a) Plot $1 + 3i$, $2 - 3i$, and $2 + 3i$ on the complex plane.

(b) Devise a graphical way to add two complex numbers. Hint: first draw arrows from the origin to the points.

(c) Draw the unit circle, and plot the points $(\sqrt{2}/2, \sqrt{2}/2)$ and $(\sqrt{3}/2, 1/2)$ on the circle. What are their polar coordinates?

(d) What happens when you multiply

$$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)?$$

Foil it out.

(e) Use $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to convert both those numbers to the form $e^{i\theta}$. Now what happens when you multiply? Use rules of exponents. Check that you got the same answer as before.

(f) Devise a general graphical way to multiply complex numbers.