Douglass Houghton Workshop, Section 2, Tue 04/02/19

Worksheet Never Gonna Give You Up

1. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.

   (a) $e^x =$

   (b) $e^{-x} =$

   (c) $\cos(x) =$

   (d) $\sin(x) =$

   (e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$

   (f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

2. The symbol i is often used to represent $\sqrt{-1}$. *It is not a real number*, because of course any real number, when squared, is positive, but $i^2 = -1$. Just the same, it is often very useful (not just in math, but in physics and engineering) to form the set of **complex numbers**

   $$\{x + iy : x \text{ and } y \text{ are real numbers}\}$$

   and then try to do with complex numbers everything we're used to doing with real numbers. (Most things will work, some won't, and some will work better.)

   (a) We know that $i^2 = -1$, so $i^3 = i^2 \cdot i = (-1) \cdot i = -i$. Write down some more powers of i until you have a general formula for i^n.

   (b) Use the power series you found in the last problem above to find $\cosh(i\theta)$, where θ is a real number.

   (c) Find $\sinh(i\theta)$.

   (d) Add them together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

   (e) Evaluate at $\theta = \pi$.

   (f) Admire your work, with wonder and amazement.

3. There is nothing special at latitude $14°38'53''$ N, longitude $78°6'28''$ W. It's just a point in the ocean. But, if you were to shoot a neutrino from the middle of the Diag (latitude $42°16'36''$ N, longitude $83°44'15''$ W) to that point, through the earth's crust, its deepest point would be directly under a very interesting place. Find that place.

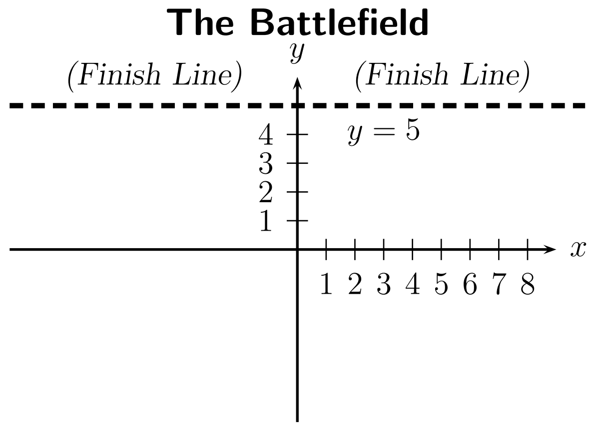
4. Consider the functions

   $$\cosh(t) = \frac{e^t + e^{-t}}{2} \qquad \text{and} \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}.$$

   (a) What shape do you get when you plot the parametric curve $x = \cos(t)$, $y = \sin(t)$? Draw it on the board, and write down an equation that relates x to y with no t's.

   (b) What do you get when you plot the parametric curve $x = \cosh(t)$, $y = \sinh(t)$? Plot some points on the board. What happens when t gets big? Try to find an equation relating x and y.

5. (This problem appeared on a Winter, 2003 Math 116 exam) The newest FOX reality show, "BattleBugs: Clash of the Beetles" begins (at $t = 0$) with eight assorted insects placed randomly on a large mat (the "battlefield", pictured here), on which is marked a "finish line". The producers hoped that the bugs would battle to be first to cross the finish line, but instead they wander around, each according to its nature. The motion of each bug is described by the equations below. Both x and y are measured in inches.



Hercules Beetle	**Ladybug**	**Tiger Beetle**	**Longhorned Beetle**
$x(t) = \cos(t/2)$ $y(t) = \sin(t/2)$	$x(t) = e^{-t}$ $y(t) = e^{-2t}$	$x(t) = 1 + t$ $y(t) = -1 + 8t$	$x(t) = 3 + t$ $y(t) = 4 - t$
Dung Beetle	**Scarab**	**June Beetle**	**African Ground Beetle**
$x(t) = t$ $y(t) = -2$	$x(t) = 2 - 7t$ $y(t) = -1 - 7t$	$x(t) = 0$ $y(t) = -1$	$x(t) = \sin(t)$ $y(t) = \cos(t)$

Which bug (or bugs)...

(a) move repetitively?

(b) begin closest to the finish line?

(c) move fastest?

(d) will move very slowly (or not at all), in the long run?

(e) will reach the finish line first?

(f) gets the dizziest?

6. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1 x + a_2 x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

where, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

Suppose $f(x) = F_0 + F_1 x + F_2 x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

(a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

(b) What happens when you add those two together? Compare with $f(x)/x$.

(c) Deduce a simple formula for $f(x)$.