Douglass Houghton Workshop, Section 2, Tue 02/26/19

Worksheet If Music Be the Food of Love, Play On

With a lot of hard work, we filled the table to the right with the values of $\int_{-\pi}^{\pi} f\,dx(x)g(x)\,dx$, where f is the row and g is the column, and m and n are positive integers.

	1	$\sin(nx)$	$\cos(nx)$
1	2π	0	0
$\sin(mx)$	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$	0
$\cos(mx)$	0	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$

The implication was that for a function of the form

$$f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots \\ + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots \tag{1}$$

integrating against a sine or cosine function makes almost all the terms 0, so for $n \geq 1$:

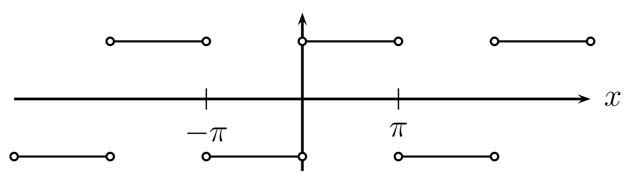
$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0, \quad \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \pi a_n, \quad \text{and} \quad \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \pi b_n.$$

1. Suppose that you have a function with the form of Equation (1), but you don't know the coefficients. You can, however, find integrals like the ones above.
   - How can you find a_1, the coefficient of $\cos(x)$?
   - How can you find a_n and b_n?

2. Consider the square wave:

   $$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ \ \ 1 & \text{if } \ \ 0 < x < \pi \end{cases}$$

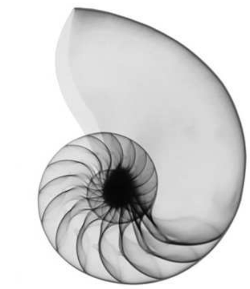
   and that pattern is repeated every 2π.

   Suppose the square wave can be written in terms of sines and cosines, as in Equation (1) above. Find the a_n and the b_n.

3. (This problem appeared on a Winter, 2003 Math 116 exam) Fred likes to juggle. So does Jason. The number of minutes Fred can juggle five balls without dropping one is a random variable, with probability density function $f(t) = 0.8e^{-0.8t}$. Similarly, the function $j(t) = 1.5e^{-1.5t}$ describes Jason's skill. Here t is time *in minutes*.
   (a) Find $\int_0^\infty f(t)\,dt$.
   (b) What percentage of Jason's juggling attempts are "embarrassing," meaning they last for 10 seconds or less?
   (c) How long can Fred juggle, on average?
   (d) Who is the better juggler? Give a good reason for your decision.

4. (6 pts) (This problem appeared on a Winter, 2003 Math 116 exam) The chambered nautilus builds a spiral sequence of closed chambers. It constructs them from the inside out, with each chamber approximately 20% larger (by volume) than the last. (The large open section at the top is not a "chamber.") The largest chamber is 9 cubic inches. Show your work on both parts.

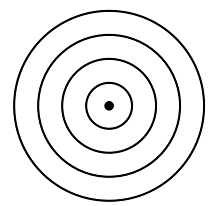


   (a) How much volume is enclosed by the last 15 chambers constructed?

   (b) How much volume is enclosed by *all* the chambers? Assume for simplicity that there are infinitely many chambers.

5. Find the probability of winning the pass bet in craps.

6. Consider a game of "continuous darts". The board is circular, as you expect, with radius 1. The goal is to get as close to the middle as possible. If a dart lands a distance r from the bullseye, its score is $1 - r$. (So every number between 0 and 1 is a possible score.)



   A novice player throws a dart which lands randomly somewhere on the board. That means that for any region R on the board,

$$\text{Prob(dart lands in } R) = \frac{\text{area of } R}{\text{area of board}}.$$

   (a) Fill in the table with the probabilities that the dart scores **below** the given value.

x	0	1/4	1/2	3/4	1
Prob(score $< x$)					

   (b) Let x be any number. Find the probability that the score is less than x.

   (c) Find the median score.

7. The function you found in (6b) above is called the **cumulative distribution function** or **CDF** of the score. Let's call it $P(x)$.

   (a) Use $P(x)$ to find the probability that the score is between 1/3 and 2/3.

   (b) How would you use $P(x)$ to find the probability that a score is between a and b?

   (c) Hmmm, that answer reminds you of the First Fundamental Theorem of Calculus, I bet. Can you write it as an integral?

   The derivative of $P(x)$ is called the **probability density function** or **PDF** of the score. Let's call if $p(x)$.

8. Find the mean score of Continuous Darts by computing the integral

$$\int_{-\infty}^{\infty} xp(x)\,dx.$$