Douglass Houghton Workshop, Section 1, Wed 04/10/19

Worksheet Never Was So Much Owed By So Many To So Few

1. Previously we found formulas for converting from latitude (ϕ) and longitude (θ) to Cartesian coordinates:

$$x = \rho \cos\phi \cos\theta$$
$$y = \rho \cos\phi \sin\theta$$
$$z = \rho \sin\phi$$

Where the origin is the center of the earth and ρ is the distance from the center, which is 1 on the earth's surface.

   (a) How can you find ρ if you know x, y, and z? (Hint: in polar coordinates, how do you find r in terms of x and y?)

   (b) Likewise find formulas for ϕ and θ in terms of x, y, and z.

2. There is *still* nothing special at latitude 14°38′53″ N, longitude 78°6′28″ W. It's just a point in the ocean. But, if you were to shoot a neutrino from the middle of the Diag (latitude 42°16′36″ N, longitude 83°44′15″ W) to that point, through the earth's crust, its deepest point would be directly under a very interesting place. Find that place, using the Google spreadsheet linked on our homepage.

3. Last time we found a remarkable power series:

$$\frac{x}{1-x-x^2} = x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+21x^8+34x^9+55x^{10}+\cdots = \sum_{n=0}^{\infty} F_n x^n$$

where F_n is the nth Fibonacci number, defined by $F_n = F_{n-1} + F_{n-2}$. That's already a pretty cool result. But if we play our cards right, we can parlay it into a non-recursive formula for the Fibonacci numbers.

   (a) If a is a constant, what is the power series for $\frac{1}{1-ax}$ about $x = 0$?

   (b) Verify that if $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ then $(1-\alpha x)(1-\beta x) = 1 - x - x^2$.

   (c) Now suppose we could split the generating function above like this:

$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

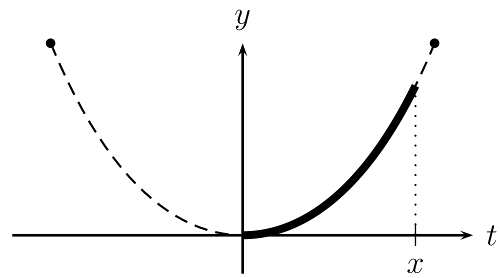
   for some constants A and B. Find what A and B must be to make the equation above work for all values of x.

   (d) Now find the series for $\frac{A}{1-\alpha x}$ and $\frac{B}{1-\beta x}$, in Σ form, and add them together to get a formula for the Fibonacci numbers.

4. We've made some progress finding the shape of a hanging chain. If the shape is given by $F(x)$, then by considering forces and arc length we've shown that

$$F''(x) = \frac{\delta g}{T_0}\sqrt{1 + F'(x)^2}$$

where T_0 is the tension at the bottom of the chain, δ is the mass density of the chain, and g is acceleration due to gravity (all constants). Where to go from here? We'd like to find a formula for $F(x)$.

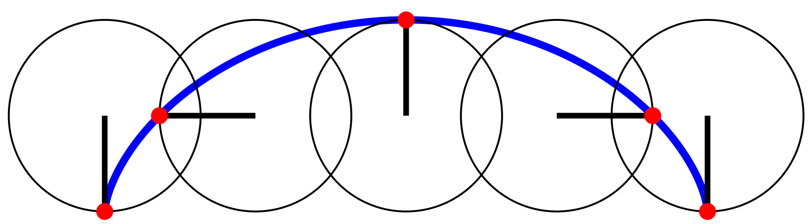


(a) Hmmm. No Fs, only F's. And lots of constants. Let $y = F'(x)$, and put all the constants together into one constant. That should make it look better.

(b) What is y when x is 0? Now you have an initial value to go with your differential equation.

(c) Separate the variables and solve the differential equation.

5. Tommy is still riding a unicycle in the dark. The only thing you can see is a red reflector which is on the outer edge of the wheel, which has radius r. As Tommy moves from left to right, the reflector traces out the path below:



Progress: Previously we found a table of values for the reflector's position:

t	0 sec	1/4 sec	1/2 sec	3/4 sec	1 sec
x_c = center's x	0	$\pi r/2$	πr	$3\pi r/2$	$2\pi r$
y_c = center's y	0	0	0	0	0
x_r = reflector's x	0	$\pi r/2 - r$	πr	$3\pi r/2 + r$	$2\pi r$
y_r = reflector's y	$-r$	0	r	0	$-r$

(a) Find formulas for x_c, y_c, x_r, and y_r in terms of t. HINT on x_r: Add a line to the table for $x_r - x_c$.

(b) Find the exact distance traveled by the reflector in one minute. Solve by hand. No approximations! HINT: $\cos 2\theta = 1 - 2\sin^2\theta$.

6. (Adapted from a Winter, 2010 exam problem)

(a) Find the first four nonzero terms of the Taylor series for $\ln(1 + x)$ about $x = 0$.

(b) Find the first three nonzero terms of the Taylor series for $g(x) = \ln\left(\frac{1+x}{1-x}\right)$ about $x = 0$. Hint: Rules of logarithms.

(c) Find the exact value of the sum of the series $2\left(\frac{3}{4}\right) + \frac{2}{3}\left(\frac{3}{4}\right)^3 + \frac{2}{5}\left(\frac{3}{4}\right)^5 + \cdots$