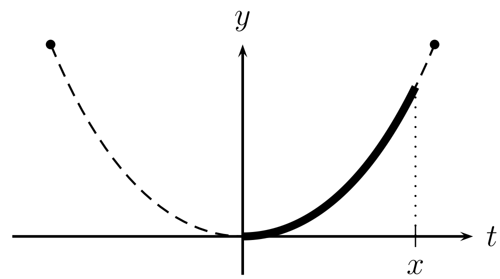
Douglass Houghton Workshop, Section 1, Mon 04/08/19

Worksheet May the Road Rise to Meet You

1. In our quest to determine the shape of a hanging chain, we have found that the forces on a portion of the chain obey a certain relationship: if $m(x)$ is the mass of the chain between the middle and position x, T_0 is the tension in the chain at the bottom, and $y = F(x)$ is the shape of the chain, then in order to make the forces balance we must have:

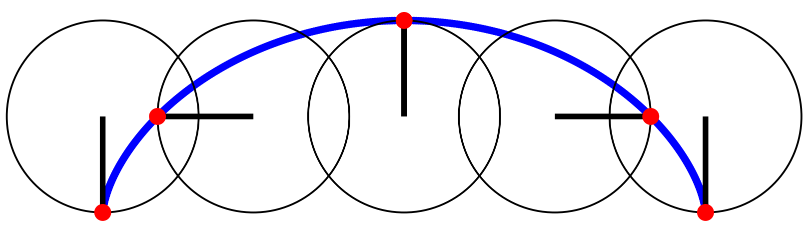
$$\frac{m(x)g}{T_0} = F'(x).$$



   (a) How could you calculate $m(x)$ if you knew $F(x)$?

   (b) Some of that we know how to do. Use it to modify the equation above. Feel free to combine unknown constants into one, and simplify as much as possible.

2. Suppose you are watching Tommy ride a unicycle in the dark. The only thing you can see is a red reflector which is on the outer edge of the front wheel. As Tommy moves from left to right, you see the reflector trace out the path below:



   (a) The wheels in the picture represent snapshots taken every quarter second. The dots are the reflector. Assuming the wheel's center in the first snapshot is $(0, 0)$ and the radius of the wheel is r, fill in the table below with the center's position and the reflector's position at time t.

t	0 sec	1/4 sec	1/2 sec	3/4 sec	1 sec
x_c = center's x	0				
y_c = center's y	0				
x_r = reflector's x					
y_r = reflector's y					

   (b) Find formulas for x_c, y_c, x_r, and y_r in terms of t.

   (c) Find the exact distance traveled by the reflector in one minute. No approximations!

3. Last time we found that latitudes and longitudes can't be averaged to find a midpoint. This left us bitter and disillusioned, but undaunted. If we could just convert to (x, y, z) coordinates, then we'd be in business.

   Write ϕ for latitude and θ for longitude. Define the (x, y, z) coordinate system as:

   - The origin is at the center of the earth.
   - The radius of the earth has length 1.
   - The x axis goes through the point $(\phi = 0, \theta = 90\text{W})$, near the Galapogos Islands.
   - The y axis goes through the point $(\phi = 0, \theta = 0)$, off the coast of Nigeria.
   - The z axis goes through the North Pole.

   (a) Find z in terms of ϕ and θ. (One of them doesn't matter.)

   (b) Now find x and y. Hint: the plane at latitude ϕ intersects the earth in a circle. Draw it on the board. What is its radius?

4. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1x + a_2x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

   where, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

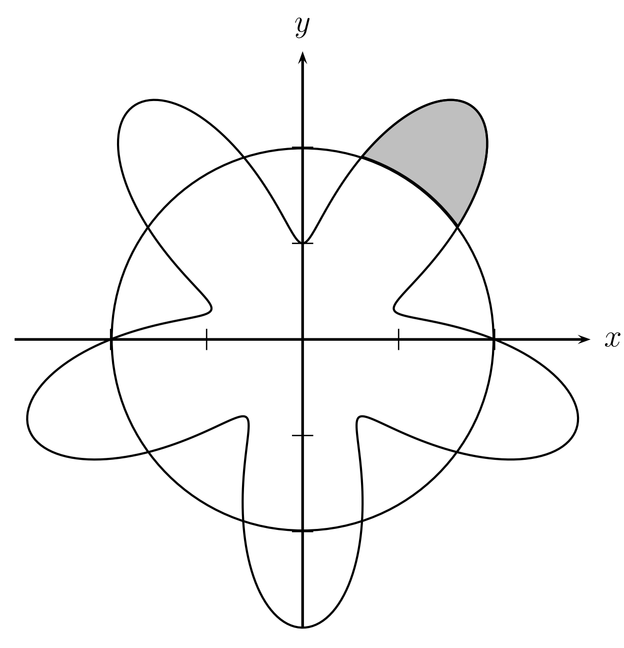
   Suppose $f(x) = F_0 + F_1x + F_2x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

   (a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

   (b) What happens when you add those two together? Compare with $f(x)/x$.

   (c) Deduce a simple formula for $f(x)$.

5. (Adapted from a Fall, 2010 Math 116 Exam) In the picture to the right, the graphs of $r = 2$ and $r = 2 - \sin(5\theta)$ are shown.



   (a) Write a definite integral that computes the shaded area.

   (b) Compute the area exactly.

   (c) Write an integral for the length of the boundary of the shaded area.

   (d) Get an approximate answer for that length, using your calculator.