Douglass Houghton Workshop, Section 1, Mon 04/01/19

Worksheet Kimono

1. Last week we talked about the origins of the World Wide Web, and posed the question of how Google was able, in 1998, to be so much better than its competition. For reasons unexplained, we considered a family frisbee game, where 5 people (Mom, Dad, their young son Octavius, his 15-year-old sister Amy, and her new boyfriend Jeff) throw to each other, but according to their own biases:

   - Mom throws to everyone else with probability $\frac{1}{4}$.
   - Dad throws to his two children with probability $\frac{1}{2}$ each.
   - Octavius can't throw very far, so he throws to his two neighbors, Dad and Amy.
   - Amy always throws to Jeff.
   - Jeff, futilely trying to win approval, throws to Mom and Dad with probability $\frac{1}{2}$ each.

   We're interested in who has the frisbee most often in the long run.

   (a) On the right side of your board make a 5×5 table whose rows and columns are named for the players. Use abbreviations M,D,O,A,J. In (row i, column j) put the probability that person i throws to person j.

   (b) To the left of that, make a 5 column table, whose rows are labeled $0, 1, 2, \cdots$ and whose columns represent the players. In this table we'll put the probability that each player has the frisbee after some number of throws.

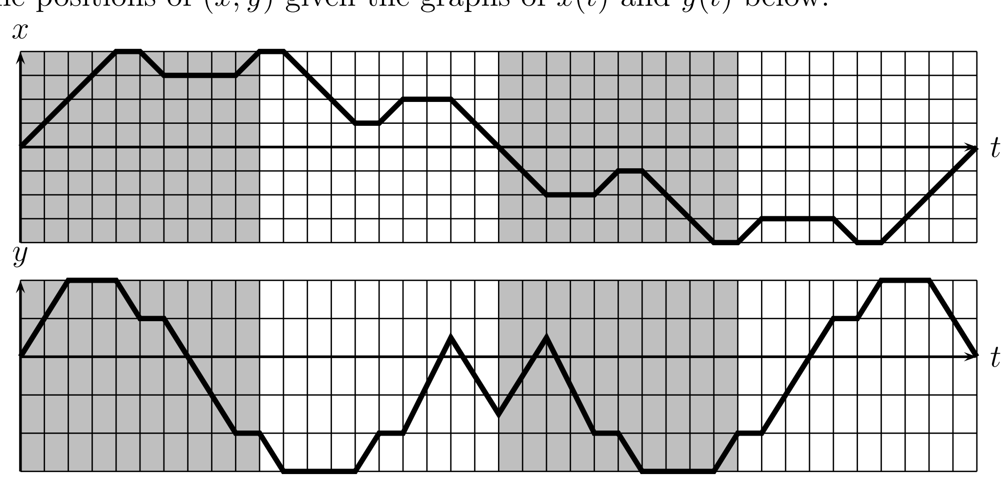
   (c) Mom starts with the frisbee, so put $1, 0, 0, 0, 0$ in the 0^{th} row.

   (d) In the next row fill in the probabilities after 1 throw. This shouldn't be hard.

   (e) Now it gets interesting. What are the ways that Dad could have the frisbee after 2 throws? Find the total probability that he does and put that in row 2, column D. Likewise for the other players.

   (f) How could you fill in row 3?

2. Plot the positions of (x, y) given the graphs of $x(t)$ and $y(t)$ below:



3. On April first, Brooke and Spencer like to play practical jokes on people. Their rule, of course, is that every practical joke done on them demands an equal and opposite retaliation. From year to year they keep a mental "balance sheet" that records how much grief they "owe" or are "owed" by their rivals. Due to various interventions by authority, they have been very good the last few years, and they currently "owe" 100 practical jokes.

   They decide that every year, they will pay off 20% of their "debt", by playing pranks on their friends. At the same time, their friends play 5 pranks on them each year.

   (a) What will the "balance" be at the end of 4/1/2019?

   (b) Fill in the table with the balance B_n at the end of $4/1/(2018+n)$.

n	0	1 (2019)	2 (2020)	3	4	5
B_n	100					

   (c) Find a formula for B_n in terms of n.

   (d) What happens in the long run? (Does the sequence $B_0, B_1, B_2, \ldots$ converge?)

4. Find the full Taylor series for $f(x) = \dfrac{1}{\sqrt{1-4x}}$ about $x = 0$. Also find the radius of convergence.

5. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.

   (a) $e^x =$

   (b) $e^{-x} =$

   (c) $\cos(x) =$

   (d) $\sin(x) =$

   (e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$

   (f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

6. The symbol i is often used to represent $\sqrt{-1}$. *It is not a real number*, because of course any real number, when squared, is positive, but $i^2 = -1$. Just the same, it is often very useful (not just in math, but in physics and engineering) to form the set of **complex numbers**

$$\{x + iy : x \text{ and } y \text{ are real numbers}\}$$

   and then try to do with complex numbers everything we're used to doing with real numbers. (Most things will work, some won't, and some will work better.)

   (a) We know that $i^2 = -1$, so $i^3 = i^2 \cdot i = (-1) \cdot i = -i$. Write down some more powers of i until you have a general formula for i^n.

   (b) Use the power series you found in the last problem above to find $\cosh(i\theta)$, where θ is a real number.

   (c) Find $\sinh(i\theta)$.

   (d) Add them together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

   (e) Evaluate at $\theta = \pi$.

   (f) Admire your work, with wonder and amazement.