Douglass Houghton Workshop, Section 1, Mon 02/25/19

Worksheet Go Placidly Amid the Noise and Haste

With a lot of hard work, we filled the table to the right with the values of $\int_{-\pi}^{\pi} f\,dx(x)g(x)\,dx$, where f is the row and g is the column, and m and n are positive integers.

	1	$\sin(nx)$	$\cos(nx)$
1	2π	0	0
$\sin(mx)$	0	$\begin{cases} \pi & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$	0
$\cos(mx)$	0	0	$\begin{cases} \pi & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$

1. Let $h(x) = 5 + 2\cos(x) + \sin(x) - 5\cos(2x) + 3\sin(2x)$.

   (a) Use your calculator to compute:

   $$\int_{-\pi}^{\pi} h(x)\,dx =$$
   $$\int_{-\pi}^{\pi} h(x)\cos(x)\,dx =$$
   $$\int_{-\pi}^{\pi} h(x)\sin(x)\,dx =$$
   $$\int_{-\pi}^{\pi} h(x)\cos(2x)\,dx =$$
   $$\int_{-\pi}^{\pi} h(x)\sin(2x)\,dx =$$

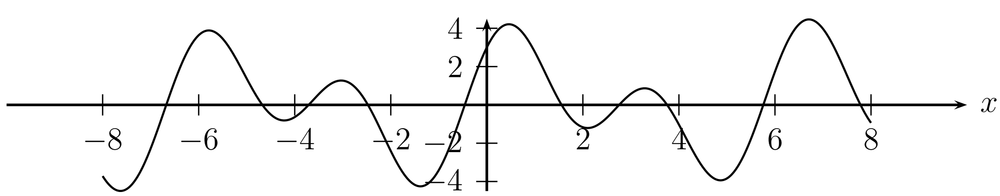
   (b) Explain the results using the table above.

2. Predict what the integrals in (1a) above will be if we change $h(x)$ to

   $$h(x) = 2 + 3\cos(x) - 7\sin(x) - 4\cos(2x) + \sin(2x).$$

3. Generalize: What will those integrals be if

   $$h(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x).$$

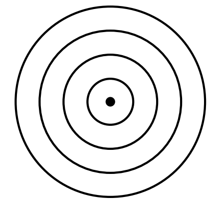
4. 

   Suppose $h(x)$ is some function you measure in nature, and its graph looks like the one above. You do some numercal integration and discover that

   $$\int_{-\pi}^{\pi} h(x)\,dx = 0 \qquad \int_{-\pi}^{\pi} h(x)\cos(2x)\,dx = 6.28$$
   $$\int_{-\pi}^{\pi} h(x)\cos(x)\,dx = 3.14 \qquad \int_{-\pi}^{\pi} h(x)\sin(2x)\,dx = 4.71$$
   $$\int_{-\pi}^{\pi} h(x)\sin(x)\,dx = 6.28 \qquad \int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = \int_{-\pi}^{\pi} h(x)\sin(nx)\,dx = 0 \text{ for } n \geq 3.$$

   Can you guess a formula for $h(x)$? Use what you know, and check by graphing your formula on a calculator.

5. Consider a game of "continuous darts". The board is circular, as you expect, with radius 1. The goal is to get as close to the middle as possible. If a dart lands a distance r from the bullseye, its score is $1-r$. (So every number between 0 and 1 is a possible score.)



A novice player throws a dart which lands randomly somewhere on the board. That means that for any region R on the board,

$$\text{Prob(dart lands in } R) = \frac{\text{area of } R}{\text{area of board}}.$$

(a) Fill in the table with the probabilities that the dart scores **below** the given value.

x	0	1/4	1/2	3/4	1
Prob(score $< x$)					

(b) Let x be any number. Find the probability that the score is less than x.

(c) Find the median score.

6. The function you found in (5b) above is called the **cumulative distribution function** or **CDF** of the score. Let's call it $P(x)$.

(a) Use $P(x)$ to find the probability that the score is between 1/3 and 2/3.

(b) How would you use $P(x)$ to find the probability that a score is between a and b?

(c) Hmmm, that answer reminds you of the First Fundamental Theorem of Calculus, I bet. Can you write it as an integral?

The derivative of $P(x)$ is called the **probability density function** or **PDF** of the score. Let's call if $p(x)$.

7. Find the mean score of Continuous Darts by computing the integral

$$\int_{-\infty}^{\infty} xp(x)\,dx.$$

8. Find the probability of winning the pass bet in craps.

9. (6 pts) (This problem appeared on a Winter, 2003 Math 116 exam) The chambered nautilus builds a spiral sequence of closed chambers. It constructs them from the inside out, with each chamber approximately 20% larger (by volume) than the last. (The large open section at the top is not a "chamber.") The largest chamber is 9 cubic inches. Show your work on both parts.

(a) How much volume is enclosed by the last 15 chambers constructed?

(b) How much volume is enclosed by *all* the chambers? Assume for simplicity that there are infinitely many chambers.