Douglass Houghton Workshop, Section 2, Thu 4/11/2017

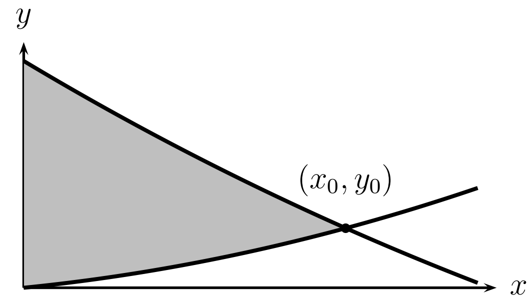
Worksheet So Long, and Thanks For All The Fish

1. Last week we found a recurrence for the D_n, the number of successful Secret Santa games with n players:
$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$
Since there are $n!$ total ways for n people to pick names, the probability of a successful game is $P_n = D_n/n!$.
   (a) Make a recurrence for P_n using the fact that $D_n = n!P_n$. Simplify.
   (b) Now we need to do something clever to solve the recurrence. Here's an idea: consider the value $R_n = P_n - P_{n-1}$. Find a recurrence for R_n, and solve it.
   (c) Find P_n. What is the limit of P_n as $n \to \infty$?

2. Last week we found the Fourier Series for x^2 on $[-\pi, \pi]$, namely
$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx).$$
It wasn't clear why we wanted to do that—it seemed silly to write x^2 in terms of an infinite sum of cosines. But now: plug in $x = \pi$ and see if you can find the value of
$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

3. Last time we found a remarkable power series:
$$\frac{x}{1-x-x^2} = x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+21x^8+34x^9+55x^{10}+\cdots = \sum_{n=0}^{\infty} F_n x^n$$
where F_n is the nth Fibonacci number, defined by $F_n = F_{n-1} + F_{n-2}$. That's already a pretty cool result. But if we play our cards right, we can parlay it into a non-recursive formula for the Fibonacci numbers.
   (a) If a is a constant, what is the power series for $\frac{1}{1-ax}$ about $x = 0$?
   (b) Verify that if $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ then $(1-\alpha x)(1-\beta x) = 1 - x - x^2$.
   (c) Now suppose we could split the generating function above like this:
$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$
for some constants A and B. Find what A and B must be to make the equation above work for all values of x.
   (d) Now find the series for $\frac{A}{1-\alpha x}$ and $\frac{B}{1-\beta x}$, in Σ form, and add them together to get a formula for the Fibonacci numbers.

4. (From the Fall, 2010 Math 116 final) The graph shows the area between the graphs of $f(x) = 6\cos(\sqrt{2x})$ and $g(x) = x^2 + x$. Let (x_0, y_0) be the intersection point between the graphs of $f(x)$ and $g(x)$.



   (a) Compute $P(x)$, the function containing the first three nonzero terms of the Taylor series about $x = 0$ of $f(x) = 6\cos(\sqrt{2x})$.

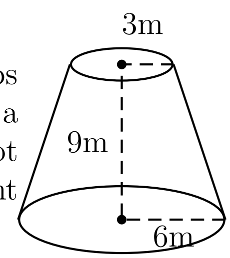
   (b) Use $P(x)$ to approximate the value of x_0.

   (c) Use $P(x)$ and the value of x_0 you computed in the previous question to write an integral that approximates the value of the shaded area. Find the value of this integral.

   (d) Graph $f(x)$ and $g(x)$ in your calculator. Use the graphs to find an approximate value for x_0.

   (e) Write a definite integral in terms of $f(x)$ and $g(x)$ that represents the value of the shaded area. Find its value using your calculator.

5. (From a Winter, 2014 Math 116 Exam) The Nub's Nob Ski Area keeps a massive supply of hot chocolate. The hot chocolate is stored in a container shaped like a cone with the point end removed. The hot chocolate has a density of 3000 kg/m^3. Recall the gravitational constant is $g = 9.8\,\text{m/s}^2$.



   (a) Write a formula for $r(h)$, the radius of a circular cross section of the container h meters above the base.

   (b) Write a formula in terms of $r(h)$ for the work required to lift a slice of hot chocolate of thickness Δh from height h to the top of the container.

   (c) Write an integral that gives the work required to lift all of the hot chocolate to the top of the container. Then evaluate the integral.

6. (Winter, 2014) Bill has just built a brand new 90,000 L swimming pool. Bill is allergic to chlorine so instead he is using a filtration system to prevent algae from building up in the pool. Algae grows in the pool at a constant rate of 600 kg/day. The filtration system receives a constant supply of 70,000 L/day of water and returns the water to the pool with 6/7ths of the algae removed. Let $A(t)$ be the amount of algae in the pool in kilograms t days after Bill has filled the pool with fresh (algae free) water.

   (a) Write down a differential equation satisfied by $A(t)$. Include an initial condition.

   (b) Solve it, and find the equilibrium solutions.