Douglass Houghton Workshop, Section 2, Thu 03/30/17

Worksheet Platypus

1. (From the Winter, 2011 Math 116 final exam) Determine whether the following series converge or diverge, and justify your answer using one or more convergence tests. No credit without justification.

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n^2}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{3n-2}{\sqrt{n^5+n^2}}$ (c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n(1+\ln(n))}$

2. (From the Winter, 2010 Math 116 Final Exam) Discover whether the following series converge or diverge, and justify using an appropriate converge test or tests. No credit without justification.

   (a) $\displaystyle\sum_{n=2}^{\infty}\frac{\sqrt{n^2+1}}{n^2-1}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n!(n+1)!}{(2n)!}$ (c) $\displaystyle\sum_{n=2}^{\infty}\frac{\sin(n)}{n^2-3}$

3. You can describe Secret Santa games in cycle notation like this: (142)(35) means person 1 gives to person 4, 4 gives to 2, and 2 gives to 1; 3 gives to 5 and 5 gives to 3. Note that (142), (421), and (214) all represent the same cycle, so to avoid confusion, we always write cycles with the lowest number first.

   A game is successful if everyone gives to someone else, i.e., no one is in a cycle by themselves. Write down all the successful games for 2, 3, and 4 people. Make a table with the number of successful games for $n = 1, 2, 3, 4$ people.

4. On April first, Alec and Benancio like to play practical jokes on people. Their rule, of course, is that every practical joke done on them demands an equal and opposite retaliation. From year to year they keep a mental "balance sheet" that records how much grief they "owe" or are "owed" by their rivals. Due to various interventions by authority, they have been very good the last few years, and they currently "owe" 100 practical jokes.

   They decide that every year, they will pay off 20% of their "debt", by playing pranks on their friends. At the same time, their friends play 5 pranks on them each year.

   (a) What will the "balance" be at the end of 4/1/2017?

   (b) Fill in the table with the balance B_n at the end of $4/1/(2016+n)$.

n	0	1 (2017)	2 (2018)	3	4	5
B_n	100					

   (c) Find a formula for B_n in terms of n. Hint: it may help to re-write the calculations above with less simplification.

   (d) What happens in the long run? (Does the sequence $B_0, B_1, B_2, \ldots$ converge?)

   (e) What if the pranks played happened continuously, instead of once a year? Write a differential equation describing that situation.

Any function which is continuous on $[-\pi, \pi]$ has a **Fourier series**. That is,

$$f(x) = A + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

for some constants $A, a_1, a_2, \ldots, b_1, b_2, \ldots$ Back before spring break we discovered how to compute the constants, namely

$$A = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

5. Let's compute the Fourier series for $f(x) = x^2$.
   (a) Compute A. (A stands for "average".)
   (b) Fill in the table to the right.
   (c) Find the a_n and b_n for $f(x) = x^2$.

n	1	2	3	4	5	6
$\sin(n\pi)$						
$\sin(-n\pi)$						
$\cos(n\pi)$						
$\cos(-n\pi)$						

6. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.
   (a) $e^x =$
   (b) $e^{-x} =$
   (c) $\cos(x) =$
   (d) $\sin(x) =$
   (e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$
   (f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

7. The symbol i is often used to represent $\sqrt{-1}$. *It is not a real number*, because of course any real number, when squared, is positive, but $i^2 = -1$. Just the same, it is often very useful (not just in math, but in physics and engineering) to form the set of **complex numbers**

$$\{x + iy : x \text{ and } y \text{ are real numbers}\}$$

   and then try to do with complex numbers everything we're used to doing with real numbers. (Most things will work, some won't, and some will work better.)
   (a) We know that $i^2 = -1$, so $i^3 = i^2 \cdot i = (-1) \cdot i = -i$. Write down some more powers of i until you have a general formula for i^n.
   (b) Use the power series you found in the last problem above to find $\cosh(i\theta)$, where θ is a real number.
   (c) Find $\sinh(i\theta)$.
   (d) Add them together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!
   (e) Evaluate at $\theta = \pi$.