Douglass Houghton Workshop, Section 2, Tue 03/28/17

Worksheet Out of the Frying Pan...

1. Last time it appeared we showed that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ Crazy!

   (a) We got that series by considering derivatives and plugging in $x = 0$. See if you can deduce a series for $\cos(x)$ the same way, by starting with

   $$\cos(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

   (b) Test it at $x = \pi$, by adding up all the terms through x^{10}. Is it close to what you expect it to be?

   (c) Do the same for $\sin(x)$.

   (d) Now systemetize the result: if we have a function $f(x)$ which has derivatives, how do we find its series? Find formulas for a_0, a_1, etc. in terms of f.

2. The integral test says that $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge. Why is this true?

   (a) Draw two copies of the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 6$ on the board.

   (b) On one graph, draw the left Riemann sum of $\int_1^{\infty} \frac{dx}{x}$ with $\Delta x = 1$. Explain why the picture shows that $\int_1^{\infty} \frac{dx}{x} < \sum_{n=1}^{\infty} \frac{1}{n}$.

   (c) Draw the right sum with $\Delta x = 1$ on the other graph. Explain why this shows that $\int_1^{\infty} \frac{dx}{x} > \sum_{n=2}^{\infty} \frac{1}{n}$.

   (d) How does this prove the integral test for $f(x) = 1/x$?

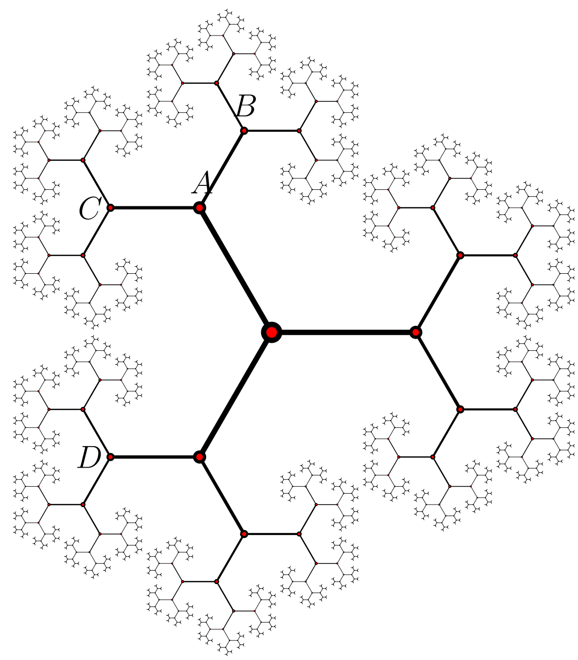
3. Between 1980 and 2009, there were 7015 Division I basketball games which were tied at the end of regulation play, resulting in at least one overtime period. Suppose p is the probability that an overtime period ends in a tie.

   (a) How many 2nd overtimes do you expect there were? How many 3rd overtimes? How many kth overtimes?

   (b) There were in fact 8568 total overtime periods between 1980 and 2009. Use that to estimate p.

   (c) About how many games went 6 or more overtimes, do you guess?

4. The figure below contains only 120° angles. As you move away from the center, the line segments get shorter by a factor r. That is, the longest segments (connected to the center) have length 1, the next longest have length r, the next longest after that have length r^2, etc. Assume the branches keep splitting and splitting, ad infnitum. Most of your answers will be in terms of r, but we'll be able to find what r is in part (h). No '...' or '$\sum$' allowed in any of your answers.



(a) Suppose you start at the center and follow the generally northward path. That is, go to A, then turn right and go to B, then turn left, right, left, etc. How far will you travel after n steps? How far will you travel if you take an infnite number of steps?

(b) If you take the path described in part (a), how far to the north will you have gone when you reach A? (That is, how much higher on the page is A than the center?) How far north will you have gone when you reach B? When you have gone n steps?

(c) Use the result of part (b) to give the total height of the figure.

(d) Find the distance from the center to the left side of the figure by following the generally northwestward path that goes to A, then turns left to C, then right, left, right, etc. This time you want the horizontal distance travelled.

(e) Find the distance from the center to the right side of the figure by following a generally northeastward path. Thus find the total width of the figure.

Now of course, the picture could be drawn with any value of r. But if r were too large, the figure would overlap itself, and if r were too small, it would hard to see what's going on. The picture above was drawn by using the largest possible value of r which doesnt cause overlap. Thus the path that goes generally southward from C never crosses the path that goes generally northward from D, but they do approach the same point.

(f) Find the vertical distance from C to D by using a path through the center.

(g) Find the same distance by considering the southward path from C and the northward path from D.

(h) Set them equal and solve for r. Do you recognize this number?