Douglass Houghton Workshop, Section 1, Mon 04/10/17

Worksheet Play it Again, Sam

1. Last time we found a recurrence for the D_n, the number of successful Secret Santa games with n players:
$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$
Since there are $n!$ total ways for n people to pick names, the probability of a successful game is $P_n = D_n/n!$.

   (a) Make a recurrence for P_n using the fact that $D_n = n!P_n$. Simplify as much as possible.

   (b) Now we need to do something clever to solve the recurrence. Here's an idea: consider the value $R_n = P_n - P_{n-1}$. Find a recurrence for R_n, and solve it.

   (c) Find P_n. What is the limit of P_n as $n \to \infty$?

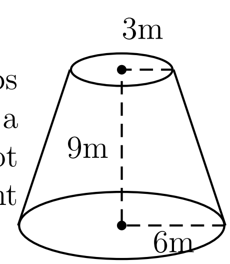
2. (This problem appeared on a Fall, 2004 Math 116 exam.)

   (a) Find the second order Taylor polynomial of $f(x) = \sqrt{4+x}$ for x near 0.

   (b) Find the Taylor series about $x = 0$ of $\sin(2x)$, either from scratch or by using a series you know already.

   (c) Using your answers to parts (a) and (b) and *without computing any derivatives*, find the second order Taylor polynomial that approximates $g(x) = \sqrt{4 + \sin(2x)}$ for x near 0.

   (d) The error in a Taylor polynomial approximation is mostly in the first term omitted, in this case the third order term. So compute that, and give an estimate of the maximum error in the approximation when $-0.1 \leq x \leq 0.1$.



3. (From a Winter, 2014 Math 116 Exam) The Nub's Nob Ski Area keeps a massive supply of hot chocolate. The hot chocolate is stored in a container shaped like a cone with the point end removed. The hot chocolate has a density of $3000\,\mathrm{kg/m^3}$. Recall the gravitational constant is $g = 9.8\,\mathrm{m/s^2}$.

   (a) Write a formula for $r(h)$, the radius of a circular cross section of the container h meters above the base.

   (b) Write a formula in terms of $r(h)$ for the work required to lift a slice of hot chocolate of thickness Δh from height h to the top of the container.

   (c) Write an integral that gives the work required to lift all of the hot chocolate to the top of the container. Then evaluate the integral.

4. Long, long ago we found formulas for converting from latitude (ϕ) and longitude (θ) to Cartesian coordinates:

$$\begin{aligned} x &= \cos\phi\cos\theta \\ y &= \cos\phi\sin\theta \\ z &= \sin\phi \end{aligned}$$

Here the origin is the center of the earth, and the radius of the earth is 1. The only catch was that the formulas assume that a point is on the surface of the earth.

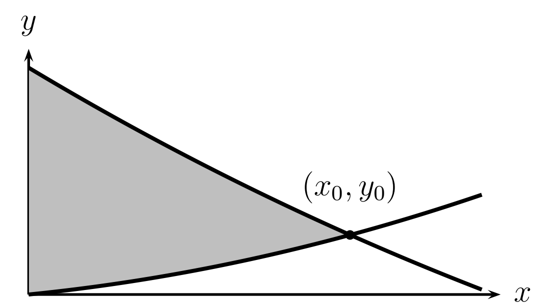
(a) How far is the point $P = (x, y, z)$ from the origin?

(b) If you multiply all the coordinates of P by the same number, you get a point that is either directly above or directly below P. Suppose P is below the surface of the earth. What are the Cartesian coordinates of the point on the surface directly above P?

(c) Find a way to convert the Cartesian coordinates of a point on the surface back to latitude and longitude.

5. There is *still* nothing special at latitude 14°38′53″ N, longitude 78°6′28″ W. It's just a point in the ocean. But, if you were to shoot a neutrino from the middle of the Diag (latitude 42°16′36″ N, longitude 83°44′15″ W) to that point, through the earth's crust, its deepest point would be directly under a very interesting place. Find that place, using the Google spreadsheet linked on our homepage.

6. (From the Fall, 2010 Math 116 final) The graph shows the area between the graphs of $f(x) = 6\cos(\sqrt{2x})$ and $g(x) = x^2 + x$. Let (x_0, y_0) be the intersection point between the graphs of $f(x)$ and $g(x)$.



(a) Compute $P(x)$, the function containing the first three nonzero terms of the Taylor series about $x = 0$ of $f(x) = 6\cos(\sqrt{2x})$.

(b) Use $P(x)$ to approximate the value of x_0.

(c) Use $P(x)$ and the value of x_0 you computed in the previous question to write an integral that approximates the value of the shaded area. Find the value of this integral.

(d) Graph $f(x)$ and $g(x)$ in your calculator. Use the graphs to find an approximate value for x_0.

(e) Write a definite integral in terms of $f(x)$ and $g(x)$ that represents the value of the shaded area. Find its value using your calculator.