Douglass Houghton Workshop, Section 1, Mon 04/03/17

Worksheet Never Say Never

1. (From the Winter, 2010 Math 116 Final Exam) Discover whether the following series converge or diverge, and justify using an appropriate converge test or tests. No credit without justification.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}}{n^2-1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{\sin(n)}{n^2-3}$

2. You can describe Secret Santa games in cycle notation like this: (142)(35) means person 1 gives to person 4, 4 gives to 2, and 2 gives to 1; 3 gives to 5 and 5 gives to 3. Note that (142), (421), and (214) all represent the same cycle, so to avoid confusion, we always write cycles with the lowest number first.

   A game is successful if everyone gives to someone else, i.e., no one is in a cycle by themselves. Write down all the successful games for 2, 3, and 4 people. Make a table with the number of successful games for $n = 1, 2, 3, 4$ people.

Any function which is continuous on $[-\pi, \pi]$ has a **Fourier series**. That is,

$$f(x) = A + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

for some constants $A, a_1, a_2, \ldots, b_1, b_2, \ldots$ Back before spring break we discovered how to compute the constants, namely

$$A = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

3. Let's compute the Fourier series for $f(x) = x^2$.
   (a) Compute A. (A stands for "average".)
   (b) Fill in the table to the right.
   (c) Find the a_n and b_n for $f(x) = x^2$.

n	1	2	3	4	5	6
$\sin(n\pi)$ $\sin(-n\pi)$ $\cos(n\pi)$ $\cos(-n\pi)$						

4. (From the Fall, 2013 Math 116 Final Exam) Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{2^n}{3n}(x-5)^n.$$

   and rigorously justify you answer, explaining what convergence test(s) you use and how you used them.

5. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.

   (a) $e^x =$

   (b) $e^{-x} =$

   (c) $\cos(x) =$

   (d) $\sin(x) =$

   (e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$

   (f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

6. The symbol i is often used to represent $\sqrt{-1}$. *It is not a real number*, because of course any real number, when squared, is positive, but $i^2 = -1$. Just the same, it is often very useful (not just in math, but in physics and engineering) to form the set of **complex numbers**

   $$\{x + iy : x \text{ and } y \text{ are real numbers}\}$$

   and then try to do with complex numbers everything we're used to doing with real numbers. (Most things will work, some won't, and some will work better.)

   (a) We know that $i^2 = -1$, so $i^3 = i^2 \cdot i = (-1) \cdot i = -i$. Write down some more powers of i until you have a general formula for i^n.

   (b) Use the power series you found in the last problem above to find $\cosh(i\theta)$, where θ is a real number.

   (c) Find $\sinh(i\theta)$.

   (d) Add them together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

   (e) Evaluate at $\theta = \pi$.

7. (Winter, 2013) A boat's initial value is \$100,000; it loses 15% of its value each year. The boat's maintenance cost is \$500 the first year and increases by 10% annually.

   (a) Let B_n be the value of the boat n years after it was purchased. Find B_1 and B_2.

   (b) Find a formula for B_n.

   (c) Let M_n be the total amount of money spent on the maintenance in the first n years. Find M_2 and M_3.

   (d) Find a closed form formula for M_n.