Douglass Houghton Workshop, Section 1, Mon 02/20/17

Worksheet Hotline Bling

We've done a lot of integrals with sines and cosines, and mostly filled in this table with the results of $\int_{-\pi}^{\pi} f(x)g(x)\,dx$.

$f \setminus g$	1	$\sin(nx)$	$\cos(nx)$
1	2π	0	0
$\sin(mx)$	0		0
$\cos(mx)$	0	0	

Here are some trig identities you should know:

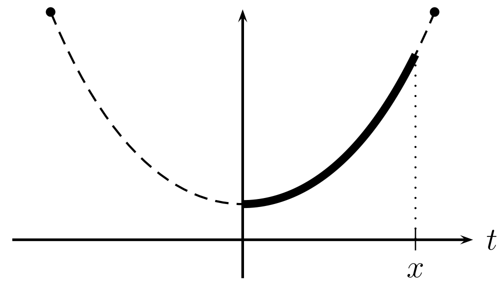
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \qquad \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \qquad \sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

1. Use the identities above to find a formula for $\cos(\alpha)\cos(\beta)$ that looks easier to integrate.
2. Compute $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx$, either by using your last result, or by using parts.
3. Compute $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx$. Look for a shortcut.
4. Let $h(x) = 5 + \sin(x) + 2\cos(x) + 3\sin(2x) - 5\cos(2x)$.
   (a) Use your calculator to compute:
   $$\int_{-\pi}^{\pi} h(x)\,dx = \qquad \int_{-\pi}^{\pi} h(x)\sin(2x)\,dx =$$
   $$\int_{-\pi}^{\pi} h(x)\sin(x)\,dx = \qquad \int_{-\pi}^{\pi} h(x)\cos(2x)\,dx =$$
   $$\int_{-\pi}^{\pi} h(x)\cos(x)\,dx =$$
   (b) Explain the results using the table above.
5. Predict what the integrals in (4a) above will be if we change $h(x)$ to
   $$h(x) = 2 + 3\sin(x) - 7\cos(x) - 4\sin(2x) + \cos(2x).$$
6. Generalize: What will those integrals be if
   $$h(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + b_1\sin(x) + b_2\sin(2x).$$
7. If $f(x)$ is any positive-valued function, what is the derivative of $\ln(f(x))$?
8. If $f(x) = \sec(x) + \tan(x)$, what is $f'(x)$?
9. So what is $\dfrac{d}{dx}\ln(\sec(x) + \tan(x))$?

10. We've made some progress finding the shape of a hanging chain. If the shape is given by $F(x)$, then by considering forces and arc length we've shown that

$$F''(x) = \frac{\delta g}{T_0}\sqrt{1 + F'(x)^2}$$

where T_0 is the tension at the bottom of the chain, δ is the mass density of the chain, and g is acceleration due to gravity (all constants). Where to go from here? We'd like to find a formula for $F(x)$.



  (a) Hmmm. No Fs, only F's. And lots of constants. Let $y = F'(x)$, and put all the constants together into one constant. That should make it look better.

  (b) What is y when x is 0? Now you have an initial value to go with your diffeq.

  (c) Separate the variables and solve the differential equation.

11. Last time we found formulas for converting from latitude (ϕ) and longitude (θ) to Cartesian coordinates:

$$\begin{aligned} x &= \cos\phi\cos\theta \\ y &= \cos\phi\sin\theta \\ z &= \sin\phi \end{aligned}$$

Here the origin is the center of the earth, and the radius of the earth is 1. The only catch was that the formulas assume that a point is on the surface of the earth.

  (a) How far is the point $P = (x, y, z)$ from the origin?

  (b) If you multiply all the coordinates of P by the same number, you get a point that is either directly above or directly below P. Suppose P is below the surface of the earth. What are the Cartesian coordinates of the point on the surface directly above P?

  (c) Find a way to convert the Cartesian coordinates of a point on the surface back to latitude and longitude.

12. There is *still* nothing special at latitude $14°38'53''$ N, longitude $78°6'28''$ W. It's just a point in the ocean. But, if you were to shoot a neutrino from the middle of the Diag (latitude $42°16'36''$ N, longitude $83°44'15''$ W) to that point, through the earth's crust, its deepest point would be directly under a very interesting place. Find that place, using Excel.

13. Find the probability of winning the "Pass" bet in craps.