Douglass Houghton Workshop, Section 2, Tue 4/10/2012

Worksheet To Infinity, and Beyond

1. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1x + a_2x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	1	1	2	3	5	8	13	21	34	55

where, for $n >= 2$, $F_n = F_{n-1} + F_{n-2}$.

Suppose $f(x) = F_0 + F_1x + F_2x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

(a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

(b) What happens when you add those two together? Compare with $f(x)/x$.

(c) Deduce a simple formula for $f(x)$.

2. Last time we found these Taylor series:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

Let $i = \sqrt{-1}$. We found some powers of i last week.

n	i^n	n	i^n	n	i^n	n	i^n
0	1	1	i	2	-1	3	$-i$
4	1	5	i	6	-1	7	$-i$
8	1	9	i	10	-1	11	$-i$

(a) Use the power series we found last time to find $\cosh(i\theta)$, where θ is a real number.

(b) Find $\sinh(i\theta)$.

(c) Of course $\cosh(x) + \sinh(x) = e^x$, so add the last two answers together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

(d) Evaluate at $\theta = \pi$.

3. (A simple model of photosynthesis of individual leaves) Photosynthesis is a complex mechanism; the following model is a very simplified charicature. Suppose that a leaf contains a number of traps that can capture light. If a trap captures light, it becomes energized. The energy in the trap can then be used to produce sugar, which causes the energized trap to become unenergized. The number of traps that can become energized is proportional to the the number of unenergized traps and the light intensity. Denote by T the total number of traps (unenergized and energized) in a leaf, by I the light intensity, and by x the number of energized traps. Then the following differential equation describes how the number of energized traps changes over time:

$$\frac{dx}{dt} = k_1(T - x)I - k_2x$$

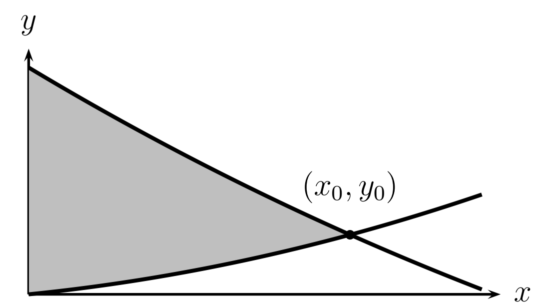
where k_1 and k_2 are positive constants. Find all the equilibria and study their stability.

4. (From the Fall, 2011 Math 116 Final Exam) When a voltage V in volts is applied to a series circuit consisting of a resistor with resistance R in ohms and an inductor with inductance L, the current $I(t)$ at time t is given by

$$I(t) = \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \qquad \text{where } V,\ R, \text{ and } L \text{ are constants.}$$

   (a) Show that $I(t)$ satisfies $\dfrac{dI}{dt} = \dfrac{V}{L}\left(1 - \dfrac{R}{V}I\right)$.
   (b) Find a Taylor series for $I(t)$ about $t = 0$. Write the first three nonzero terms and a general term of the Taylor series.
   (c) Use the Taylor series to compute $\lim_{t\to 0} \dfrac{I(t)}{t}$.

5. (From the Fall, 2010 Math 116 final) The graph shows the area between the graphs of $f(x) = 6\cos(\sqrt{2x})$ and $g(x) = x^2 + x$. Let (x_0, y_0) be the intersection point between the graphs of $f(x)$ and $g(x)$.



   (a) Compute $P(x)$, the function containing the first three nonzero terms of the Taylor series about $x = 0$ of $f(x) = 6\cos(\sqrt{2x})$.
   (b) Use $P(x)$ to approximate the value of x_0.
   (c) Use $P(x)$ and the value of x_0 you computed in the previous question to write an integral that approximates the value of the shaded area. Find the value of this integral.
   (d) Graph $f(x)$ and $g(x)$ in your calculator. Use the graphs to find an approximate value for x_0.
   (e) Write a definite integral in terms of $f(x)$ and $g(x)$ that represents the value of the shaded area. Find its value using your calculator.

6. On the Mercury Project memorial plaque in Florida we say this equation:

$$v = R_0\sqrt{\frac{g}{R_0 + h}}$$

which gave the speed of an object in orbit at height h.

   (a) Find the 2^{nd} degree Taylor approximation around $a = 0$ of v as a function of h.
   (b) Use part (a) with $R_0 = 6371\,\text{km}$ and $g = 9.8\,\text{m/s}^2$ to approximate the speed of the International Space Station, which is at an altitude of 340 km.