Douglass Houghton Workshop, Section 2, Thu 4/5/2012

Worksheet So Long, and Thanks for All the Fish

1. We know by the integral test that

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

converges. But what does it converge to?

(a) Use your calculator to find the first dozen or so partial sums. Can you guess what the limit is? If you like, type in the calculator program on the right and let it run, to see how the partial sums change.

```
0 → S
1 → N
Lbl 10
S+1/N² → S
N+1 → N
Disp S
Goto 10
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(b) Last class we found that the Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$ is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx).$$

Plug in $x = \pi$ and see if you can find $\zeta(2)$.

Aside: The sum is called $\zeta(2)$ because it's a value of the famous *Riemann zeta function*, defined by

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Fully understanding the Riemann zeta function is one of the seven "Millenium Problems" proposed by the Clay Mathematics Institute in 2000. It is worth a prize of one million dollars to the first person to do it.

2. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1 x + a_2 x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	1	1	2	3	5	8	13	21	44	65

where, for $n >= 2$, $F_n = F_{n-1} + F_{n-2}$.

Suppose $f(x) = F_0 + F_1 x + F_2 x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

(a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

(b) What happens when you add those two together? Compare with $x^2 f(x)$.

(c) Deduce a simple formula for $f(x)$.

3. Last time we found these Taylor series:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

The symbol i is often used to represent the imaginary number $\sqrt{-1}$. Last time we found that

$$i^n = \begin{cases} 1 & \text{if } n = 0, 4, 8, 12, \ldots \\ i & \text{if } n = 1, 5, 9, 13, \ldots \\ -1 & \text{if } n = 2, 6, 10, 14, \ldots \\ -i & \text{if } n = 3, 7, 11, 15, \ldots \end{cases}$$

(a) Use the power series we found last time to find $\cosh(i\theta)$, where θ is a real number.

(b) Find $\sinh(i\theta)$.

(c) Of course $\cosh(x) + \sinh(x) = e^x$, so add the last two answers together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

(d) Evaluate at $\theta = \pi$.

4. (From the Fall, 2004 Math 116 Final Exam) We shall investigate a well-known physical phenomenon, called the "Doppler Effect". When an electromagnetic signal (e.g. a ray of light) with frequency F_e is emitted from a source moving away with velocity $v > 0$ with respect to a receiver at rest, then the received frequency F_r is different from F_e. The relationship linking the emitted frequency F_e and the received frequency F_r is the Doppler Law:

$$F_r = \sqrt{\frac{1 - v/c}{1 + v/c}} F_e, \qquad \text{where } c \text{ is a constant, the speed of light.}$$

For this problem, you might find useful to know that the third order Taylor polynomial for the function $\sqrt{\frac{1+x}{1-x}}$ near $x = 0$ is $1 + x + \frac{x^2}{2} + \frac{x^3}{2}$.

(a) On Earth, nearly all objects travel with velocities v much smaller than the speed of light c, i.e. the ratio v/c is very small. Use this fact to obtain the Doppler Law for slow-moving emitters:

$$F_r \approx \left(1 - \frac{v}{c}\right) F_e.$$

(b) The relationship in part (a) is not exact, and an error is made when it is used to approximate the Doppler Law. Find an expression for the "error", in terms of v, c and F_e. Is the approximation accurate within 1% of F_e if the velocity is at most 10% of the speed of light c? Explain.

5. Compute the Taylor series for $\frac{1}{\sqrt{1-4x}}$. Then find the interval of convergence.