Douglass Houghton Workshop, Section 2, Tue 4/3/2012

Worksheet Rastafarian

1. On April first, Molli and Natalie like to play practical jokes on people. Their rule, of course, is that every practical joke done on them demands an equal and opposite retaliation. From year to year they keep a mental "balance sheet" that records how much grief they "owe" or are "owed" by their rivals. Due to various interventions by authority, they have been very good the last few years, and they currently "owe" 100 practical jokes.

   They decide that every year, they will pay off 20% of their "debt", by playing pranks on their friends. At the same time, their friends play 5 pranks on them each year.

   (a) What will the "balance" be at the end of 4/1/2012?

   (b) Fill in the table with the balance B_n at the end of $4/1/(2011+n)$.

n	0	1 (2012)	2 (2013)	3	4	5
B_n	100					

   (c) Find a formula for B_n in terms of n.

   (d) What happens in the long run? (Does the sequence $B_0, B_1, B_2, \ldots$ converge?)

   (e) What if the pranks played happened continuously, instead of once a year. Write a differential equation describing that situation.

Any function which is continuous on $[-\pi, \pi]$ has a **Fourier series**. That is,

$$f(x) = A + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

for some constants $A, a_1, a_2, \ldots, b_1, b_2, \ldots$ Back before spring break we discovered how to compute the constants, namely

$$A = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

2. Let's compute the Fourier series for $f(x) = x^2$.

   (a) Compute A. (A stands for "average".)

   (b) Fill in the table to the right.

   (c) Find the a_n and b_n for $f(x) = x^2$.

n	1	2	3	4	5	6
$\sin(n\pi)$						
$\sin(-n\pi)$						
$\cos(n\pi)$						
$\cos(-n\pi)$						

3. (From a Fall, 2008 Math 116 exam) Find the interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{n}$.

4. Find the full Taylor series for $f(x) = \dfrac{1}{\sqrt{1-4x}}$ about $x = 0$. Also find the interval of convergence.

5. (Adapted from a Winter, 2010 exam problem) Last week we found the Taylor series for $\ln(1+x)$ about $x = 0$.

   (a) Write down the first four nonzero terms of the series for $\ln(1+x)$, either from memory or by working them out.

   (b) Find the first three nonzero terms of the Taylor series for $g(x) = \ln\left(\dfrac{1+x}{1-x}\right)$ about $x = 0$. Hint: Rules of logarithms.

   (c) Find the exact value of the sum of the series

$$2\left(\frac{3}{4}\right) + \frac{2}{3}\left(\frac{3}{4}\right)^3 + \frac{2}{5}\left(\frac{3}{4}\right)^5 + \cdots$$

6. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.

   (a) $e^x =$

   (b) $e^{-x} =$

   (c) $\cos(x) =$

   (d) $\sin(x) =$

   (e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$

   (f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

7. The symbol i is often used to represent $\sqrt{-1}$. *It is not a real number*, because of course any real number, when squared, is positive, but $i^2 = -1$. Just the same, it is often very useful (not just in math, but in physics and engineering) to form the set of **complex numbers**

$$\{x + iy : x \text{ and } y \text{ are real numbers}\}$$

   and then try to do with complex numbers everything we're used to doing with real numbers. (Most things will work, some won't, and some will work better.)

   (a) We know that $i^2 = -1$, so $i^3 = i^2 \cdot i = (-1) \cdot i = -i$. Write down some more powers of i until you have a general formula for i^n.

   (b) Use the power series you found in the last problem above to find $\cosh(i\theta)$, where θ is a real number.

   (c) Find $\sinh(i\theta)$.

   (d) Add them together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

   (e) Evaluate at $\theta = \pi$.