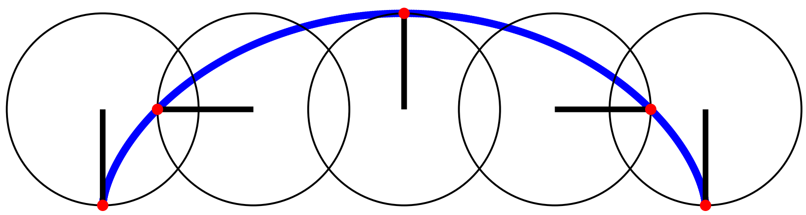
Douglass Houghton Workshop, Section 2, Thu 3/8/2012

Worksheet Never Say Never

1. Find the probability of winning the "Pass" bet in craps.

2. Shannon is still riding a unicycle in the dark. The only thing you can see is a red reflector which is on the outer edge of the wheel, which has radius r. As Shannon moves from left to right, the reflector traces out the path below:



Progress: Last time we showed that the parametric equations describing the motion of the reflector are

$$x = 2\pi rt - r\sin(2\pi t) \qquad \text{and} \qquad y = -r\cos(2\pi t)$$

so the distance travelled by the reflector in 1 second is

$$\begin{aligned} L &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{(2\pi r - 2\pi r\cos(2\pi t))^2 + (2\pi r\sin(2\pi t))^2}\,dt \\ &= 2\pi r\int_0^1 \sqrt{2 - 2\cos(2\pi t)}\,dt. \end{aligned}$$

Finish finding the exact length (no calculators). Recall that $\cos 2\theta = 1 - 2\sin^2\theta$.

3. (This problem appeared on a Winter, 2009 Math 116 exam) The quantity

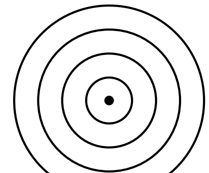
$$\int_1^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$$

roughly models the resistance that football-shaped plankton encounter when falling through water. Note that $a = 1$, $b = 2$, and $c = 3$ are constants that describe the dimensions of the plankton. Find a value of M for which

$$\int_1^M \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$$

differs from the original model of resistance by at most 0.001. Hint: make use of the integral and the comparison test.

4. Consider a game of "continuous darts". The board is circular, as you expect, with radius 1. The goal is to get as close to the middle as possible. If a dart lands a distance r from the bullseye, its score is $1 - r$. (So every number between 0 and 1 is a possible score.)



A novice player throws a dart which lands randomly somewhere on the board. That means that for any region R on the board,

$$\text{Prob(dart lands in } R) = \frac{\text{area of } R}{\text{area of board}}.$$

(a) Fill in the table with the probabilities that the dart scores **below** the given value.

x	0	1/4	1/2	3/4	1
Prob(score $< x$)					

(b) Let x be any number. Find a formula for the probability that the score is less than x. This is the *cumulative distribution function* (CDF).

(c) Find the mean and median scores.

5. A spaceship seeks to reach a height y above the surface of the earth. The force of gravity at any time is

$$F_g = G\frac{Mm}{r^2}$$

where

$$\begin{aligned} G &= \text{The universal gravitational constant} \\ M &= \text{The mass of the earth} \\ m &= \text{The mass of the spaceship} \\ r &= \text{The distance from the spaceship to the center of the earth.} \end{aligned}$$

(a) How much work will it take to raise the spaceship to from the surface of the earth to a point y meters above the surface? Use R for the radius of the earth. Don't assume gravity is constant as the ship moves upward!

(b) How much work would it take to push the spaceship entriely beyond the reach of Earth's gravity? (Let $y \to \infty$.)

(c) If the ship is travelling at velocity v, it will have kinetic energy $\frac{1}{2}mv^2$. That energy will be converted into work to move the ship upward. What speed must the ship be going near the surface to leave the earth's gravity well? This is the earth's *escape velocity*.

(d) Look up the values of G, M, and R, and get a numerical answer in miles per second.