Douglass Houghton Workshop, Section 2, Tue 2/21/2012

Worksheet Kowabunga

With a lot of hard work, we filled in the table to the right with the values of $\int_{-\pi}^{\pi} f(x)g(x)\,dx$, where f is the row and g is the column.

	1	$\sin(nx)$	$\cos(nx)$
1	2π	0	0
$\sin(mx)$	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$	0
$\cos(mx)$	0	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$

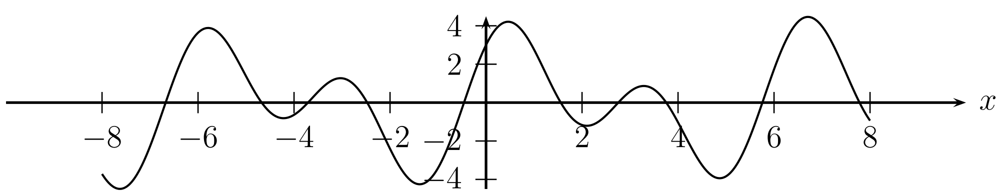
The implication was that for a function of the form

$$\begin{aligned} f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots \\ + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots \end{aligned}$$

integrating against a sine or cosine function makes almost all the terms 0, so

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0, \quad \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \pi a_n, \quad \text{and} \quad \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \pi b_n.$$

1. Consider the function $h(x)$ whose graph looks like this:



   Suppose you do some numerical integration and discover that

$$\begin{aligned} &\int_{-\pi}^{\pi} h(x)\,dx = 0 && \int_{-\pi}^{\pi} h(x)\cos(2x)\,dx = 6.28 \\ &\int_{-\pi}^{\pi} h(x)\cos(x)\,dx = 3.14 && \int_{-\pi}^{\pi} h(x)\sin(2x)\,dx = 4.71 \\ &\int_{-\pi}^{\pi} h(x)\sin(x)\,dx = 6.28 && \int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = \int_{-\pi}^{\pi} h(x)\sin(nx)\,dx = 0 \text{ for } n \geq 3. \end{aligned}$$

   Can you guess a formula for $h(x)$?

2. Help Barbie to become pink! If t is a number between 0 and 1 which represents the current grayness of a pixel (0 = black, 1 = white), come up with three functions $r(t)$, $g(t)$, and $b(t)$ which give the red, green, and blue values for what the pixel *should* look like.

3. Last time we guessed correctly that $V_C = R_0\sqrt{\dfrac{g}{R_0 + h}}$ is the velocity needed to achieve a circular orbit at height h above the surface of the earth, where

$$R_0 = \text{the radius of the earth } (6371\,\text{km}), \text{ and}$$
$$g = \text{the acceleration due to gravity } (9.8\,\text{m/s}^2).$$

   (a) So how long does it take to orbit the earth at height h?

   (b) How high would you have to be in order for it to take 24 hours to make one orbit?

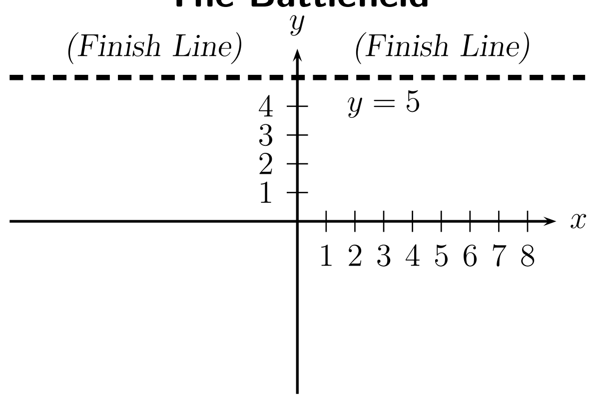
Remember what we found when computing the probabilitiy of the "Hard Eight" bet in craps:

$$\text{Prob of winning in } n \text{ rolls} = W + WC + WC^2 + \cdots + WC^{n-1} = W\frac{1 - C^n}{1 - C}$$
$$\text{Prob of winning} = W + WC + WC^2 + WC^3 + \cdots = \frac{W}{1 - C}.$$

4. Find the probability of winning the "Pass" bet in craps.

5. (This problem appeared on a Winter, 2003 Math 116 exam) The newest FOX reality show, "BattleBugs: Clash of the Beetles" begins (at $t = 0$) with eight assorted insects placed randomly on a large mat (the "battlefield", pictured here), on which is marked a "finish line". The producers hoped that the bugs would battle to be first to cross the finish line, but instead they wander around, each according to its nature. The motion of each bug is described by the equations below. Both x and y are measured in inches.

The Battlefield



Hercules Beetle $x(t) = \cos(t/2)$ $y(t) = \sin(t/2)$	**Ladybug** $x(t) = e^{-t}$ $y(t) = e^{-2t}$	**Tiger Beetle** $x(t) = 1 + t$ $y(t) = -1 + 8t$	**Longhorned Beetle** $x(t) = 3 + t$ $y(t) = 4 - t$
Dung Beetle $x(t) = t$ $y(t) = -2$	**Scarab** $x(t) = 2 - 7t$ $y(t) = -1 - 7t$	**June Beetle** $x(t) = 0$ $y(t) = -1$	**African Ground Beetle** $x(t) = \sin(t)$ $y(t) = \cos(t)$

Which bug (or bugs)...

   (a) move repetitively?

   (b) move fastest?

   (c) begin closest to the finish line?

   (d) will reach the finish line first?

   (e) will move very slowly (or not at all), in the long run?