Douglass Houghton Workshop, Section 2, Thu 1/19/2012

Worksheet Et tu, Brute?

1. There is still nothing special at latitude 14°38′53″ N, longitude 78°6′28″ W. It's just a point in the ocean. But, if you were to shoot a neutrino from the middle of the Diag (latitude 42°16′36″ N, longitude 83°44′15″ W) to that point, through the earth's crust, its deepest point would be directly under a very interesting place. Find that place, and how deep the neutrino is there.

   Progress:
   - Averaging the latitudes and longitudes doesn't work, but
   - If we could find the cartesian coordinates for the two points, then we could average them and get the answer.

2. Let's practice some integration by parts.

   (a) $\int xe^x \, dx$

   (b) $\int \ln x \, dx$

   (c) $\int e^x \sin x \, dx$

   (d) $\int_0^1 \tan^{-1}(x) \, dx$ Hint: $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$

3. Last time we found two ways to show that

$$\int_{-\pi}^{\pi} \sin(mt) \cos(mt) \, dt = 0.$$

   First because sin is odd and cos is even, their product is odd, so the integral must be 0. The other way used the angle sum formulas for sin:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

   to get $\sin(mt) \cos(nt)$ in terms of something we could integrate.

   (a) Let m and n be positive integers. Think of a similar trick to approach

$$\int_{-\pi}^{\pi} \sin(mt) \sin(nt) \, dt.$$

   (b) Evaluate the integral assuming $m \neq n$.

   (c) Do the same for the case $m = n$.

4. Consider the **gamma function**: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$, for $x > 0$.

   (a) Use integration by parts to prove that $\Gamma(x+1) = x\Gamma(x)$.

   (b) Show that $\Gamma(1) = 1$. Then fill in this chart, using part (a):

x	1	2	3	4	5	6
$\Gamma(x)$						

   (c) So if x is a positive integer, what is $\Gamma(x)$?

5. Last time we found that the number of ways to arrange m white beads and n black beads is an entry in Pascal's triangle. Some notation:

$$\binom{n}{k} = \text{the } k\text{th entry in the } n\text{th row of Pascal's triangle}$$

   where rows and columns are numbered starting at 0. It's pronounced "n choose k".

   (a) Suppose you play 10 games of roulette, betting on red each time. At the end you see that you have broken even. But you can't remember the order in which you won or lost, due to all the excitement. How many different ways could you have broken even, and what was the probability of each?

   (b) So what was the probability that you broke even?

6. (This problem appeared on a Fall, 2005 math 116 exam.) Circle City is a circular city with a radius of five miles. A straight highway runs East-West through the center of the city. The density of the population at a distance y (in miles North or South) from the road is well approximated by

$$D(y) = 4y$$

   (in thousands of people per square mile.) Apparently many people in Circle City like to live as far from the highway as possible.

   (a) Write a Riemann sum that approximates the total population of Circle City.

   (b) Write an integral that gives the total population of Circle City.

   (c) Evaluate your integral to find the total population of Circle City.

7. Suppose you have a function $f(x)$. You know:

   - f is a quadratic. That is, $f(x) = ax^2 + bx + c$ for some constants a, b, and c.
   - How to measure $f(-1)$, $f(0)$, and $f(1)$.

   You want to know $\int_{-1}^{1} f(x)\, dx$.

   (a) Let R, S, and T be the values you measure for $f(-1)$, $f(0)$, and $f(1)$, respectively. Find a formula for $f(x)$. (That is, find a, b, and c in terms of R, S, and T.)

   (b) Find $\int_{-1}^{1} f(x)\, dx$ in terms of the R, S, and T.