Douglass Houghton Workshop, Section 1, Mon 4/9/2012

Worksheet Questions for the Ages

1. We know by the integral test that

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

converges. But what does it converge to?

   (a) Use your calculator to find the first dozen or so partial sums. Can you guess what the limit is? If you like, type in the calculator program on the right and let it run, to see how the partial sums change.

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0 → S
1 → N
Lbl 10
S+1/N² → S
N+1 → N
Disp S
Goto 10
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   (b) Last class we found that the Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$ is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx).$$

   Plug in $x = \pi$ and see if you can find $\zeta(2)$.

Aside: The sum is called $\zeta(2)$ because it's a value of the famous *Riemann zeta function*, defined by

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Fully understanding the Riemann zeta function is one of the seven "Millenium Problems" proposed by the Clay Mathematics Institute in 2000. It is worth a prize of one million dollars to the first person to do it.

2. Last time we found these Taylor series:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

Let $i = \sqrt{-1}$. We found some powers of i last time.

n	i^n	n	i^n	n	i^n	n	i^n
0	1	1	i	2	-1	3	$-i$
4	1	5	i	6	-1	7	$-i$
8	1	9	i	10	-1	11	$-i$

   (a) Use the power series we found last time to find $\cosh(i\theta)$, where θ is a real number.

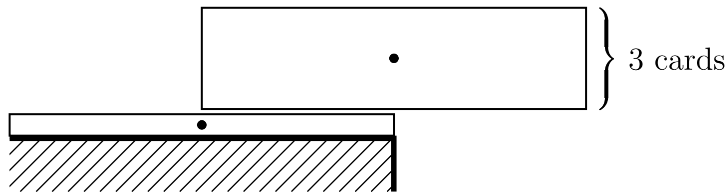
   (b) Find $\sinh(i\theta)$.

   (c) Of course $\cosh(x) + \sinh(x) = e^x$, so add the last two answers together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

   (d) Evaluate at $\theta = \pi$.

3. Last week we made a three card stack from the bottom up. We put down one card, and pushed it out halfway, until it almost fell. Then we put a card under that, flush with the table, and pushed the stack out 1/4 card length. Then we put a third card under the first two, again flush with the table, and pushed the stack out 1/6 card length.

   If we add a fourth card, how far can we push out the stack? The picture to the right may be useful. The dots show the centers of mass of the old stack and the new card.



4. Last time we found a remarkable Taylor series:

$$\frac{1}{1-x-x^2} = 1+x+2x^2+3x^3+5x^4+8x^5+13x^6+21x^7+34x^8+55x^9+\cdots = \sum_{n=0}^{\infty} F_n x^n$$

   where F_n is the nth Fibonacci number. That's already a pretty cool result. But if we play our cards right, we can parlay it into a non-recursive formula for the Fibonacci numbers.

   (a) If a is a constant, what is the power series for $\frac{1}{x-a}$ about $x=0$?

   (b) Find the roots of $1-x-x^2$.

   (c) If we call those numbers a and b, then the generating function above is

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1-x-x^2} = \frac{-1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

   for some constants A and B. Find A and B.

   (d) Now find the series for $\frac{A}{x-a}$ and $\frac{B}{x-b}$, and add them together to get a formula for the Fibonacci numbers.

5. (This problem appeared on a Fall, 2004 Math 116 exam.)

   (a) Find the second order Taylor polynomial of the function $f(x) = \sqrt{4+x}$ for x near 0.

   (b) Find the Taylor series about $x=0$ of $\sin(2x)$, either from scratch or by using a series you know already.

   (c) Using your answers to parts (a) and (b) and *without computing any derivatives*, find the second order Taylor polynomial that approximates $g(x) = \sqrt{4+\sin(2x)}$ for x near 0.

   (d) The error in a Taylor polynomial approximation is mostly in the first term omitted, in this case the third order term. So compute that, and give an estimate of the maximum error in the approximation when $-0.1 \le x \le 0.1$.