Douglass Houghton Workshop, Section 1, Wed 4/4/2012

Worksheet Pontifications

Any function which is continuous on $[-\pi, \pi]$ has a **Fourier series**. That is,

$$f(x) = A + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

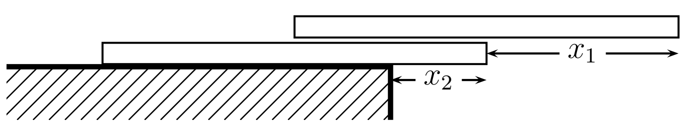
for some constants $A, a_1, a_2, \ldots, b_1, b_2, \ldots$ Back before spring break we discovered how to compute the constants, namely

$$A = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

1. Let's compute the Fourier series for $f(x) = x^2$.
   (a) Compute A. (A stands for "average".)
   (b) Fill in the table to the right.
   (c) Find the a_n and b_n for $f(x) = x^2$.

n	1	2	3	4	5	6
$\sin(n\pi)$						
$\sin(-n\pi)$						
$\cos(n\pi)$						
$\cos(-n\pi)$						

2. We're interested in how far you can get a playing card to stick out over the edge of a table. We know there must be some limit, but what is it? With two cards, we found
   (A) The center of mass of the top card must be over the bottom card, or else it will fall off. The best we can do is take it right to the edge.
   (B) The center of mass of both cards must be over the table, or else they will both fall off together. Again, the best we can do is put it right at the edge.



   (a) If the cards in the picture are extended as far as possible, find x_1 and x_2.
   (b) Put a third card under the first two, flush with the edge of the table. How far can you push the stack out? Hint: find the center of mass of the stack.

3. (Adapted from a Winter, 2010 exam problem)
   (a) Find the first four nonzero terms of the Taylor series for $\ln(1+x)$ about $x = 0$.
   (b) Find the first three nonzero terms of the Taylor series for $g(x) = \ln\left(\frac{1+x}{1-x}\right)$ about $x = 0$. Hint: Rules of logarithms.
   (c) Find the exact value of the sum of the series $2\left(\frac{3}{4}\right) + \frac{2}{3}\left(\frac{3}{4}\right)^3 + \frac{2}{5}\left(\frac{3}{4}\right)^5 + \cdots$

4. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1x + a_2x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	1	1	2	3	5	8	13	21	44	65

where, for $n >= 2$, $F_n = F_{n-1} + F_{n-2}$.

Suppose $f(x) = F_0 + F_1x + F_2x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

(a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

(b) What happens when you add those two together? Compare with $x^2f(x)$.

(c) Deduce a simple formula for $f(x)$.

5. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.

(a) $e^x =$

(b) $e^{-x} =$

(c) $\cos(x) =$

(d) $\sin(x) =$

(e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$

(f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

6. The symbol i is often used to represent $\sqrt{-1}$. *It is not a real number*, because of course any real number, when squared, is positive, but $i^2 = -1$. Just the same, it is often very useful (not just in math, but in physics and engineering) to form the set of **complex numbers**

$$\{x + iy : x \text{ and } y \text{ are real numbers}\}$$

and then try to do with complex numbers everything we're used to doing with real numbers. (Most things will work, some won't, and some will work better.)

(a) We know that $i^2 = -1$, so $i^3 = i^2 \cdot i = (-1) \cdot i = -i$. Write down some more powers of i until you have a general formula for i^n.

(b) Use the power series you found in the last problem above to find $\cosh(i\theta)$, where θ is a real number.

(c) Find $\sinh(i\theta)$.

(d) Add them together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

(e) Evaluate at $\theta = \pi$.

7. (From a Fall, 2008 Math 116 exam) Find the interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(x-1)^n}{n}$.